Implementation in Models of Independent, Private, and Multivariate Values*

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Abstract

We consider the problem of implementation in models of independent private values in which the valuation an agent attributes to a particular alternative is a function from a multidimensional Euclidean space to the real line. We first consider implementation by standard mechanisms, that include a decision rule and a profile of personal transfers. We present impossibility results on the implementation of decision rules that assign different outcomes to profiles of signals that result in the same profile of valuations. We then consider implementation by extended mechanisms that include, in addition to a decision rule and a profile of personal transfers, a profile of functions that affect the arguments of the valuation functions. We show that decision rules that assign different outcomes to profiles of signals that result in the same profile of valuations can be implemented by such mechanisms. (Keywords: *Dominant-strategy implementation; Bayesian implementation; Multidimensional mechanism design.*)

1 Introduction

The literature of mechanism design on models of independent private values with quasi-linear utilities has focused on implementation in models where agents' signals are simply

*I would like to thank Alex Gershkov for his guidance. I also want to thank Sergiu Hart, Ilan Kremer, Motty Perry, Tomer Siedner, Ran Weksler, and seminar participants at the Hebrew University for their valuable comments.

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the profiles of their valuations. The canonical model is as follows. There is a finite set of alternatives A, and an agent's signal is a vector in $\mathbb{R}^{|A|}$ where each coordinate represents the agent's value from a specific alternative. These models describe one of two scenarios: the first is that agents receive signals that are in fact their valuations. The second, and more plausible, scenario is that the valuations agents attribute to specific alternatives are functions of multiple signals, but the information that is payoff relevant for the designer are the agents' valuations. An expression of this approach, in the context of efficient auctions, can be found in Dasgupta and Maskin (2000):

> "Even with private values, a buyer may receive many signals. But as long as his private signals are uncorrelated with those of other buyers, his valuation (which is one-dimensional) will be a sufficient statistic for all his information. Thus, the private-values case is inherently one-dimensional."

However, there are many economic situations, even in private values settings, in which this perception is not valid, namely, when the payoff relevant information for the designer are not the agents' valuations but rather the values of the signals that compose these valuations. Here are some examples:

A board of directors needs to decide upon one of two possible allocations to maximize its firm's profits. Each allocation affects the profits of two of the firm's departments. The information about the profit each department will gain from each allocation is held by the firm's CEO. The CEO's valuation function represents a strictly convex preference relation over bundles of profits, because an egalitarian distribution of profits among the departments increases the CEO's reputation. However, the board of directors' valuation is simply the sum of the departments' profits. Therefore the decision rule that maximizes the board of directors' payoff assigns different outcomes to profiles of signals that result in the same valuation of the CEO.

A basketball franchise with a few youth teams needs to decide how to allocate the different training facilities among these teams. The franchise has two goals. The first is that the teams will win and be able to show accomplishments (e.g., reaching the playoffs) and the second is that the star players in the teams advance to become professional players. The effect of each allocation on both these factors is the private information of each team's coach. A coach's reputation, and hence his valuation, depends more on the success of the team than on the advancement of the star players, compared with the objective function of the franchise. Therefore it is also possible that two profiles of signals that result in different allocations, according to the first-best decision rule of the franchise, result in the same profile of valuations of the teams' coaches.

The faculty of the social sciences in a university is moving to a new building and the dean needs to decide how to allocate the different areas to the different departments. The dean is interested in two things: one is research achievements and the other is level of studies. The information on how an allocation impacts these factors in a certain department is the private information of the chair of the department, whose valuation function assigns different weights to these factors than the objective function of the faculty's dean. Therefore, again, profiles of signals that result in the same department chairs' valuations are assigned with different allocations according to the first-best decision rule of the faculty's dean.

In this paper we explore the possibility of implementation in models where agents' valuations are functions of multiple arguments, and where an agent's signal is the values of these arguments. We show that when we restrict our attention to *standard mechanisms*, that are composed of a decision rule and a profile of personal transfers, the implementation of decision rules that assign different outcomes to profiles of signals that result in the same profile of valuations is impossible.

Given this result we consider new ways to extend the variety of solutions to the designer's problem. We consider cases where the designer, in addition to the ability to assign personal transfers, may have means by which she can affect the arguments of the agents' valuation functions. An example of such a case is where the dean of the social sciences faculty has the ability to change the values of the level of teaching and of the level of research in some department, independently of her other decisions, by changing the funding for each argument. This allows the designer to activate more elaborate mechanisms than the ones that can be activated in the standard case. These *extended mechanisms* assign to each profile of agents' reports a decision rule, a profile of personal transfers, and a profile of functions that affect the arguments of the agents' valuation functions.

We show that the use of extended mechanisms allows for implementation of decision rules that assign different outcomes to profiles of signals that result in the same profile of valuations. Such decision rules are not implementable by standard mechanisms. We use this result to provide economic examples in which the first-best decision rule is implementable by an extended mechanism but not by a standard mechanism. Namely, in these examples the designer strictly benefits from activating extended mechanisms.

The rest of the paper is organized as follows. In the rest of this section we present the related literature. In Section 2 we present an illustrative example of the idea of the paper. In Section 3 we present the model. In Section 4 we discuss standard mechanisms, and we discuss the restrictions imposed on decision rules by the standard implementability property. In Section 5 we discuss the notion of extended mechanisms, and we show that the use of extended mechanisms enables the implementation of decision rules that are not implementable by standard mechanisms. Section 6 concludes.

Related Literature

Models of independent private values, where an agent's signal is the profile of her valuations over the social alternatives, has been dealt with extensively in the literature of mechanism design. These models can be considered as reduced forms of our model when we restrict our attention to decision rules that depend on agents' valuations. Vickrey (1961), Clarke (1971), and Groves (1973) considered the problem of a benevolent designer, whose objective is to maximize the sum of agents' utilities, and showed that implementable social choice functions that include the first-best decision rule exist. Myerson (1981) characterized the set of implementable decision rules in models where an agent receives utility from one and only one alternative, and used this characterization to solve the problem of an auctioneer of a single item who wishes to maximize her profits. Hellwig (1986) used this characterization to solve the problem of a benevolent designer who is subject to restrictions that prevent the implementation of the first-best decision rule. There have also been attempts to give simple characterizations of the set of implementable decision rules in broader settings, particularly in settings where agents receive utilities from multiple alternatives and these utilities are represented by a multidimensional signal. Roberts (1979) formulated the positive association of differences condition for domains of unrestricted signals. Rochet (1987) showed that a decision rule is implementable if and only if it is a subgradient of a convex function. Bikhchandani et al. (2006) formulated the weak monotonicity condition for order-based domains. Saks and Yu (2005) showed that this condition holds on convex domains. These characterizations result in many variations, due to some general results on revenue equivalence, in the characterization of the set of implementable social choice functions.

Another important case is that of interdependent valuations (i.e., an agent's valuation also depends on other agents' signals), where agents' signals are independent. In these types of models Dasgupta and Maskin (2000) showed that when signals are one-dimensional and satisfy a single-crossing property, efficient ex-post implementation is possible. In settings with multidimensional signals, Jeihel and Moldovanu (2001) showed that efficient decision rules cannot be implemented by the Bayes–Nash equilibrium,[1] and Jeihel et al. (2006) showed that for generic valuation functions only constant decision rules are ex-post implementable. Mezzeti (2004) showed that one can escape these impossibility results by using a two-stage mechanism, where in the first stage agents report their signals and a social alternative is chosen, and in the second stage agents are asked to report their valuations and transfers are executed. His proposed mechanism, however, has been criticized by Jeihel and Moldovanu (2006) for not being applicable.

2 Illustration

Let's revisit the case we've considered in the Introduction in which a firm's board of directors needs to choose one of two possible allocations, a or b. The CEO of the firm receives a 4-dimensional signal $\theta = \left(\left(\theta^{a,1}, \theta^{a,2}\right), \left(\theta^{b,1}, \theta^{b,2}\right)\right)$, where $\left(\theta^{y,1}, \theta^{y,2}\right)$, $y \in \{a, b\}$ are the profits that departments 1 and 2 will receive if alternative y is chosen. We assume $\theta^{y,j} \in [0, 1]$ for every $j \in \{1, 2\}$ and $y \in \{a, b\}$. We assume that the CEO's utility function is $v\,(y, \theta) + t = \sqrt{\theta^{y,1} \cdot \theta^{y,2}} + t$ for $y \in \{a, b\}$, where t is the profit of another department of the firm that is not affected by the allocation. The board's

[1] The result is of generic impossibility; there are however some non-generic cases where Bayesian implementation of the efficient rule is possible.

objective is to maximize the firm's profit, and its first-best decision rule is

$$q(\theta) = a \quad if\ and\ only\ if \quad \theta^{a,1} + \theta^{a,2} \geq \theta^{b,1} + \theta^{b,2}$$

We claim that the first-best decision rule is not implementable by a standard mechanism, i.e., a mechanism where the designer can only assign a transfer to department t. Assume to the contrary that it is. Implementability implies that the transfer function depends only on the chosen alternative, i.e., $t(\theta) = t(q(\theta))$. Consider the following types $\theta_1 = \left(\left(1, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ and $\theta_2 = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(1, \frac{1}{4}\right)\right)$. For each one of these types the valuation for a and the valuation for b equal $\frac{1}{2}$. The first-best decision rule assigns the following allocations: $q(\theta_1) = a$ and $q(\theta_2) = b$. Now implementability implies that $v(a, \theta_1) + t(a) \geq v(b, \theta_1) + t(b)$, namely, $\frac{1}{2} + t(a) \geq \frac{1}{2} + t(b)$. In addition it implies that $v(b, \theta_2) + t(b) \geq v(a, \theta_2) + t(a)$, namely, $\frac{1}{2} + t(b) \geq \frac{1}{2} + t(a)$. We get that $t(a) = t(b)$. Consider the type $\theta_3 = \left(\left(1, \frac{1}{8}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$, the decision rule assigns: $q(\theta_3) = a$. Implementability implies that $v(a, \theta_3) + t(a) \geq v(b, \theta_3) + t(b)$, namely, $\sqrt{\frac{1}{8}} + t(a) \geq \frac{1}{2} + t(b)$ a contradiction.

As was mentioned above, the use of a standard mechanism in this example corresponds to the case where the board of directors can assign a transfer only to department t. However, a more natural scenario is that the board of directors can assign transfers to all of the firm's departments. This enables the board of directors to activate an extended mechanism by which it can assign a transfer to each one of the departments as a function of the CEO's reports. We denote the transfers assigned to departments 1 and 2 by t_1 and t_2, respectively. We say that the decision rule $q(\theta)$ is implementable by an extended mechanism if there exist functions $t_1(\theta)$, $t_2(\theta)$, and $t(\theta)$ such that the CEO will always choose to report truthfully, i.e., such that for every $\theta \in [0,1]^4$

$$\theta \in \arg\max_{\hat{\theta} \in [0,1]^4} \sqrt{\left(\theta^{q(\hat{\theta}),1} + t_1(\hat{\theta})\right) \cdot \left(\theta^{q(\hat{\theta}),2} + t_2(\hat{\theta})\right)} + t(\hat{\theta})$$

We claim that the efficient decision rule is implementable by an extended mechanism. We define the transfer functions to be

$$t_1(\hat{\theta}) = \hat{\theta}^{q(\hat{\theta}),2} \qquad t_2(\hat{\theta}) = \hat{\theta}^{q(\hat{\theta}),1} \qquad t(\hat{\theta}) = -1/2 \cdot \left(\hat{\theta}^{q(\hat{\theta}),1} + \hat{\theta}^{q(\hat{\theta}),2}\right)$$

We show that this transfer scheme implements the first-best decision rule. First, note that for every $y \in \{a,b\}$ we have that

$$\theta \in \arg\max_{\hat{\theta} \in [0,1]^4} \sqrt{\left(\theta^{y,1} + \hat{\theta}^{y,2}\right) \cdot \left(\theta^{y,2} + \hat{\theta}^{y,1}\right)} - 1/2 \cdot \left(\hat{\theta}^{y,1} + \hat{\theta}^{y,2}\right)$$

This ensures that given that allocation y is chosen when the CEO reports her true signal, it is not worthwhile for her to deviate to an untrue signal that will result in the same allocation. Moreover, note that the utility of the CEO given truthful report equals

$$1/2 \cdot \left(\theta^{q(\theta),1} + \theta^{q(\theta),2}\right)$$

Since the decision rule chooses a if $\theta^{a,1} + \theta^{a,2} \geq \theta^{b,1} + \theta^{b,2}$ and b otherwise, it is not worthwhile for the CEO to deviate to an untrue signal that changes the allocation.

We see that the use of extended mechanisms in this case enables the board of directors to implement its first-best outcome, a result that would not be possible if it only used standard mechanisms.

3 The Model

We consider a society I of n agents $I = \{1,..,n\}$. There is a finite set A of social alternatives with cardinality $|A|$. Each agent i receives an independent signal $\theta_i \in \Theta_i$, where the set $\Theta_i = \prod_{k \in A} \left[\underline{\theta}_i^{k,1}, \overline{\theta}_i^{k,1}\right] \times \left[\underline{\theta}_i^{k,2}, \overline{\theta}_i^{k,2}\right] \subset \mathbb{R}_+^{2|A|}$; i.e., agent i receives a two-dimensional signal for each alternative. The signal θ_i is distributed according to a continuous density f_i, where $f_i(\theta_i) > 0$ for every $\theta_i \in \Theta_i$, and we denote the probability measure function by μ_i. Agent i's utility, given that alternative a is chosen, a signal θ_i, and a monetary transfer t_i, is $v_i(a, \theta_i) + t_i$, where $v_i(a, \theta_i) = u_i^a\left(\theta_i^{a,1}, \theta_i^{a,2}\right)$.

We assume that u_i^a is twice continuously differentiable and that $\frac{\partial u_i^a}{\partial \theta_i^{a,1}}(\theta_i) > 0$ and $\frac{\partial u_i^a}{\partial \theta_i^{a,2}}(\theta_i) > 0$ for every $\theta_i \in \Theta_i$ and every $a \in A$. The realization of θ_i is private information of agent i, while its distribution and the functional form of v_i are common knowledge. We denote $\Theta \equiv \times_{i \in I} \Theta_i$ with generic element θ, $\Theta_{-i} \equiv \times_{k \in I \setminus \{i\}} \Theta_k$ with generic element θ_{-i}, and $\Theta_i^k = \left[\underline{\theta}_i^{k,1}, \overline{\theta}_i^{k,1}\right] \times \left[\underline{\theta}_i^{k,2}, \overline{\theta}_i^{k,2}\right]$ with generic element θ_i^k.

A *decision rule* is a function $q : \Theta \to A$. Given an alternative $a \in A$ we denote $q^{-1}(a) = \{\theta \in \Theta \mid q(\theta) = a\}$. Given $i \in I$, $a \in A$, and $\theta_{-i} \in \Theta_{-i}$, we denote $q_i^{-1}(a, \theta_{-i}) = \{\theta_i \in \Theta_i \mid q(\theta_i, \theta_{-i}) = a\}$. We denote by *int S* the interior of a set S. For a given decision rule q we define $1_q^a(\theta)$ to be the indicator function of a. We define $q_i : \Theta_i \to \triangle(A)$ to be $q_i(\theta_i) = \left(E_{\theta_{-i}}\left(1_q^a(\theta)\right)\right)_{a\in A}$; namely, $q_i(\theta_i)$ is the probability distribution that agent i of type θ_i attributes to the events in A, given that the other agents report their types truthfully and given her prior distributions on their types. We denote a's coordinate of $q_i(\theta_i)$ by $q_i^a(\theta_i)$. We define $V_i : \Theta_i \to \mathbb{R}^{|A|}$ to be $V_i(\theta_i) = \left(u_i^a\left(\theta_i^{a,1}, \theta_i^{a,2}\right)\right)_{a\in A}$. A function $t_i : \Theta \to \mathbb{R}$ is called a *transfer function* and we define $\tau_i : \Theta_i \to \mathbb{R}$ to be $\tau_i(\theta_i) = E_{\theta_{-i}}(t_i(\theta))$.

A *standard social choice function* is composed of a decision rule $q(\theta)$ and a profile of transfer functions $(t_i(\theta))_{i\in I}$, one for each agent. We say that a standard social choice function $(q(\theta), t_1(\theta), ..., t_n(\theta))$ is *dominant-strategy implementable* if for every $i \in I$ and $\theta \in \Theta$ we have,

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} u_i^{q(\hat{\theta}_i, \theta_{-i})}\left(\theta_i^{q(\hat{\theta}_i, \theta_{-i}),1}, \theta_i^{q(\hat{\theta}_i, \theta_{-i}),2}\right) + t_i(\hat{\theta}_i, \theta_{-i})$$

We say that a decision rule $q(\theta)$ is *dominant-strategy implementable by a standard mechanism* if there exists a profile of functions $(t_i(\theta))_{i\in I}$ such that $(q(\theta), t_1(\theta), ..., t_n(\theta))$ is dominant-strategy implementable.

We say that a standard social choice function $(q(\theta), t_1(\theta), ..., t_n(\theta))$ is *Bayesian implementable* if for every $i \in I$ and $\theta_i \in \Theta_i$ we have

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} V_i(\theta_i) \cdot q_i(\hat{\theta}_i) + \tau_i\left(\hat{\theta}_i\right)$$

where $\cdot$ is the dot product. We say that a decision rule $q(\theta)$ is *Bayesian implementable by a standard mechanism* if there exists a profile of functions $(t_i(\theta))_{i\in I}$ such that $(q(\theta), t_1(\theta), ..., t_n(\theta))$ is Bayesian implementable.

4 Implementation by Standard Mechanisms

Given our model, we divide the set of decision rules into two subsets. The first is the subset of decision rules that depend on agents valuations, i.e., that assign the same outcome to profiles of signals that result in the same profile of valuations. Implementation of decision rules from this subset has been widely discussed in the mechanism design literature. The second subset is the complement of the first, i.e., decision rules with the property that there are at least two profiles of signals that result in the same profile of valuations that are assigned different outcomes. The following propositions show that implementability imposes serious limitations on decision rules from the second subset. We begin the analysis with dominant-strategy implementation.

Lemma 1. *Assume that $q(\theta)$ is dominant-strategy implementable; then for every $i \in I$ and $\theta_{-i} \in \Theta_{-i}$ and every $\dot{\theta}_i$ and $\ddot{\theta}_i$ in Θ_i that satisfy $q\left(\dot{\theta}_i, \theta_{-i}\right) = q\left(\ddot{\theta}_i, \theta_{-i}\right)$ we have $t_i\left(\dot{\theta}_i, \theta_{-i}\right) = t_i\left(\ddot{\theta}_i, \theta_{-i}\right)$.*

This result is immediate. Assume without loss of generality that $t_i\left(\dot{\theta}_i, \theta_{-i}\right) > t_i\left(\ddot{\theta}_i, \theta_{-i}\right)$; then $v_i\left(q(\dot{\theta}_i, \theta_{-i}), \ddot{\theta}_i\right) + t_i\left(\dot{\theta}_i, \theta_{-i}\right) > v_i\left(q(\ddot{\theta}_i, \theta_{-i}), \ddot{\theta}_i\right) + t_i\left(\ddot{\theta}_i, \theta_{-i}\right)$, in contradiction to dominant-strategy implementability.

Lemma 2. *Let $q(\theta)$ be a decision rule with the property that there exist $i \in I$, $\theta_{-i} \in \Theta_{-i}$, $a, b \in A$ with $\theta_{i,1} \in int\ q_i^{-1}(a, \theta_{-i})$, and $\theta_{i,2} \in q_i^{-1}(b, \theta_{-i})$ that satisfy $v_i(a, \theta_{i,1}) = v_i(a, \theta_{i,2})$ and $v_i(b, \theta_{i,1}) = v_i(b, \theta_{i,2})$; then $q(\theta)$ is not dominant-strategy implementable.*

Proof. Assume that $q(\theta)$ is dominant-strategy implementable. Lemma 1 implies that all $\theta_i \in q_i^{-1}(a, \theta_{-i})$ receive the same transfer $t_i(a, \theta_{-i})$, and that all $\theta_i \in q_i^{-1}(b, \theta_{-i})$ receive the same transfer $t_i(b, \theta_{-i})$. Now dominant-strategy implementability implies that $v_i(a, \theta_{i,1}) + t_i(a, \theta_{-i}) \geq v_i(b, \theta_{i,1}) + t_i(b, \theta_{-i})$ and, in addition, it implies that $v_i(a, \theta_{i,2}) + t_i(a, \theta_{-i}) \leq v_i(b, \theta_{i,2}) + t_i(b, \theta_{-i})$. Since $v_i(a, \theta_{i,2}) = v_i(a, \theta_{i,1})$ and $v_i(b, \theta_{i,2}) = v_i(b, \theta_{i,1})$ we get that $v_i(a, \theta_{i,1}) + t_i(a, \theta_{-i}) = v_i(b, \theta_{i,1}) + t_i(b, \theta_{-i})$. Now $\theta_{i,1} \in int\ q_i^{-1}(a, \theta_{-i})$. Therefore there exists $\theta_{i,3} \in q_i^{-1}(a, \theta_{-i})$ such that $v_i(a, \theta_{i,3}) < v_i(a, \theta_{i,1})$ and $v_i(b, \theta_{i,3}) = v_i(b, \theta_{i,1})$. We get that $q(\theta_{i,3}, \theta_{-i}) = a$ and $v_i(a, \theta_{i,3}) + t_i(a, \theta_{-i}) < v_i(b, \theta_{i,3}) + t_i(b, \theta_{-i})$, in contradiction to dominant-strategy implementability. □

Theorem 3. *Assume that $q(\theta)$ is dominant-strategy implementable. Let $(\theta_1, \theta_2, ..., \theta_n)$ and $\left(\theta_1^{'}, \theta_2^{'}, ..., \theta_n^{'}\right)$ be profiles of signals such that $v_i(c, \theta_i) = v_i\left(c, \theta_i^{'}\right)$ for every $c \in A$ and every $i \in I$.*[2] *Then $(\theta_1, \theta_2, ..., \theta_n) \in int\, q^{-1}(a)$ implies that $q\left(\theta_1^{'}, \theta_2^{'}, ..., \theta_n^{'}\right) = a$.*

Proof. We perform a sequential substitution of signals and prove inductively that after each substitution of signals the decision rule continues to map alternative a.

Step 1: We consider the profile $\left(\theta_1^{'}, \theta_2, ..., \theta_n\right)$ and claim that $q\left(\theta_1^{'}, \theta_2, ..., \theta_n\right) = a$ and that $\theta_2 \in int\, q_2^{-1}\left(a, \left(\theta_1^{'}, \theta_3, .., \theta_n\right)\right)$.

First note that $\theta \in int\, q^{-1}(a)$ implies that $\theta_1 \in int\, q_1^{-1}(a, \theta_{-1})$; Lemma 2 then implies that $q\left(\theta_1^{'}, \theta_2, ..., \theta_n\right) = a$. Assume θ_2 belongs to the boundary of the set $q_2^{-1}\left(a, \left(\theta_1^{'}, \theta_3, .., \theta_n\right)\right)$; then in every surrounding of θ_2 there exists a signal $\theta_2^{''}$ such that $q\left(\theta_1^{'}, \theta_2^{''}, \theta_3, .., \theta_n\right) = b$ for some $b \in A \setminus \{a\}$. Since $\theta \in int\, q^{-1}(a)$ we can choose $\theta_2^{''}$ such that $\left(\theta_1, \theta_2^{''}, \theta_3, .., \theta_n\right) \in int\, q^{-1}(a)$ which implies that $\theta_1 \in int\, q_1^{-1}\left(a, \left(\theta_2^{''}, \theta_3, .., \theta_n\right)\right)$. Thus we get a contradiction to Lemma 2.

Step (k-1): We consider the profile $\left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_k, ..., \theta_n\right)$ and assume that $q\left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_k, ..., \theta_n\right) = a$ and that $\theta_k \in int\, q_k^{-1}\left(a, \left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_{k+1}, ..., \theta_n\right)\right)$.

Step (k): We consider the profile $\left(\theta_1^{'}, .., \theta_k^{'}, \theta_{k+1}, ..., \theta_n\right)$ and claim that $q\left(\theta_1^{'}, .., \theta_k^{'}, \theta_{k+1}, ..., \theta_n\right) = a$ and that $\theta_{k+1} \in int\, q_{k+1}^{-1}\left(a, \left(\theta_1^{'}, .., \theta_k^{'}, \theta_{k+2}, .., \theta_n\right)\right)$.

According to step (k-1) we have that $\theta_k \in int\, q_k^{-1}\left(a, \left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_{k+1} .., \theta_n\right)\right)$; Lemma 2 implies that $q\left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_k^{'}, \theta_{k+1}, ..., \theta_n\right) = a$.

Assume θ_{k+1} belongs to the boundary of $q_{k+1}^{-1}\left(a, \left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_k^{'}, \theta_{k+2}, .., \theta_n\right)\right)$; then in every surrounding of θ_{k+1} there exists a signal $\theta_{k+1}^{''}$ such that $q\left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_k^{'}, \theta_{k+1}^{''}, \theta_{k+2}, .., \theta_n\right) = b$ for some $b \in A \setminus \{a\}$. Since $\theta \in int\, q^{-1}(a)$ we can choose $\theta_{k+1}^{''}$ such that $\left(\theta_1, .., \theta_{k-1}, \theta_k, \theta_{k+1}^{''}, \theta_{k+2}, ..., \theta_n\right) \in int\, q^{-1}(a)$ and since $\left(\theta_1, .., \theta_{k-1}, \theta_k, \theta_{k+1}^{''}, \theta_{k+2}, ..., \theta_n\right)$ and $\left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_k, \theta_{k+1}^{''}, \theta_{k+2}, .., \theta_n\right)$ are derived from each other by (k-1) substitutions we can apply the result of step (k-1) and deduce that $\theta_k \in int\, q_k^{-1}\left(a, \left(\theta_1^{'}, .., \theta_{k-1}^{'}, \theta_{k+1}^{''}, \theta_{k+2}, .., \theta_n\right)\right)$. Thus we get a contradiction to Lemma 2. □

[2] Moreover, we assume that $\left(\theta_1^{'}, .., \theta_k^{'}, \theta_{k+1}, ..., \theta_n\right) \in int\, \Theta$ for every $1 \leq k \leq n$.

Theorem 3 says that dominant-strategy implementability implies that the decision rule assigns the same outcome to profiles of signals that result in the same profile of valuations.[3] We now proceed to describe the restrictions that are imposed on decision rules by Bayesian implementability. These restrictions are similar to the ones implied by dominant-strategy implementability, except that they occur in the interim level.

Theorem 4. *Assume that $q(\theta)$ is Bayesian implementable and that for every $i \in I$, $q_i(\theta_i)$ is continuously differentiable; then for every $i \in I$, $\theta_i \in \Theta_i$ and $a \in A$, we have that for every $b \in A$, $\frac{\partial q_i^b}{\partial \theta_i^{a,1}}(\theta_i) = 0$ if and only if $\frac{\partial q_i^b}{\partial \theta_i^{a,2}}(\theta_i) = 0$ and otherwise:*

$$\frac{\partial q_i^b(\theta_i) \big/ \partial \theta_i^{a,1}}{\partial q_i^b(\theta_i) \big/ \partial \theta_i^{a,2}} = \frac{\partial u_i^a(\theta_i) \big/ \partial \theta_i^{a,1}}{\partial u_i^a(\theta_i) \big/ \partial \theta_i^{a,2}}$$

According to the theorem Bayesian implementability implies that the interim probability assigned to the alternatives in A must remain the same along the indifference curves of the valuation functions.

The structure of the proof is as follows. Bayesian implementability implies that for every parameter θ_i the argument that will maximize agent i's expected utility will be θ_i. By converting the parameters into arguments, looking at the maximum function, and applying the envelope theorem, we get the first partial derivative of agent i's utility function when he reports truthfully. The Schwartz theorem on the equivalence of the second partial derivatives then implies the above equalities.

A corollary of Theorem 4 is that Bayesian implementability implies that the decision rule must assign the same interim probability to signals that result in the same profile of valuations.[4] We now show that signals that are assigned with the same interim probability must be assigned with the same expected transfer.

Claim 5. Assume that a decision rule $q(\theta)$ is Bayesian implementable; then for every $i \in I$ and every $\dot{\theta}_i$ and $\ddot{\theta}_i$ in Θ_i that satisfy $q_i\left(\dot{\theta}_i\right) = q_i\left(\ddot{\theta}_i\right)$, we have $\tau_i\left(\dot{\theta}_i\right) = \tau_i\left(\ddot{\theta}_i\right)$.

[3]There can be deviations but only for profiles of signals that belong to boundaries of $q^{-1}(\cdot)$.

[4]The proofs of Theorem 4 and its corollary appear in the Appendix.

This result is immediate. Assume without loss of generality that $\tau_i\left(\dot{\theta}_i\right) > \tau_i\left(\ddot{\theta}_i\right)$; then $V_i\left(\ddot{\theta}_i\right)\cdot q_i\left(\dot{\theta}_i\right)+\tau_i\left(\dot{\theta}_i\right) > V_i\left(\ddot{\theta}_i\right)\cdot q_i\left(\ddot{\theta}_i\right)+\tau_i\left(\ddot{\theta}_i\right)$, in contradiction to Bayesian implementability.

The intuition for the above results is the following. Implementability implies that the transfers must be identical for signals that result in the same outcome. On the other, it implies that the utilities of two different signals that result in the same valuations profile must be identical. These restrictions impose conditions on the transfer functions that can coexist only when signals that result in the same valuations profile are assigned the same outcome. We demonstrate this in the following example.

Example 6. *The single alternative case. We consider the case where agents gain utility only from a single alternative. Examples of such cases are the production of a public good and the allocation of a single indivisible good. In such cases an agent's expected utility is* $u\left(\theta^1,\theta^2\right)q\left(\hat{\theta}^1,\hat{\theta}^2\right)+\tau\left(\hat{\theta}\right)$.[5] *We assume that* $u\left(\theta^1,\theta^2\right)$ *and* $q\left(\theta^1,\theta^2\right)$ *are quasi-concave, strictly increasing in both arguments, and continuously differentiable. Assume that there are two types,* θ_1 *and* θ_2 *in* $\Theta\subseteq\mathbb{R}^2$*, such that* $u\left(\theta_1\right)=u\left(\theta_2\right)=\dot{u}$ *and* $q\left(\theta_1\right)\neq q\left(\theta_2\right)$*. Bayesian implementability implies that* $u\left(\theta_1\right)\cdot q\left(\theta_1\right)+\tau\left(\theta_1\right)\geq u\left(\theta_1\right)\cdot q\left(\theta_2\right)+\tau\left(\theta_2\right)$*; therefore* $u\left(\theta_1\right)\cdot\left(q\left(\theta_1\right)-q\left(\theta_2\right)\right)\geq\tau\left(\theta_2\right)-\tau\left(\theta_1\right)$*. It also implies that* $u\left(\theta_2\right)\cdot q\left(\theta_2\right)+\tau\left(\theta_2\right)\geq u\left(\theta_2\right)\cdot q\left(\theta_1\right)+\tau\left(\theta_1\right)$*; therefore* $\tau\left(\theta_2\right)-\tau\left(\theta_1\right)\geq u\left(\theta_2\right)\cdot\left(q\left(\theta_1\right)-q\left(\theta_2\right)\right)$*. Now because* $u\left(\theta_1\right)=u\left(\theta_2\right)=\dot{u}$ *we get*

$$(1)\quad \dot{u}\cdot\left(q\left(\theta_1\right)-q_i\left(\theta_2\right)\right)=\tau\left(\theta_2\right)-\tau\left(\theta_1\right)$$

Now assume that there is another pair of types, θ_3 *and* θ_4*, such that* $u\left(\theta_3\right)=u\left(\theta_4\right)=\ddot{u}$ *and* $\dot{u}\neq\ddot{u}$*. Moreover,* $q\left(\theta_3\right)=q\left(\theta_1\right)$ *and* $q\left(\theta_4\right)=q\left(\theta_2\right)$*, and so we get*

$$(2)\quad \ddot{u}\cdot\left(q\left(\theta_3\right)-q\left(\theta_4\right)\right)=\tau\left(\theta_4\right)-\tau\left(\theta_3\right)$$

Since $q\left(\theta_3\right)=q\left(\theta_1\right)$ *and* $q\left(\theta_4\right)=q\left(\theta_2\right)$*, Bayesian implementability implies*

$$(3)\quad \tau\left(\theta_2\right)-\tau\left(\theta_1\right)=\tau\left(\theta_4\right)-\tau\left(\theta_3\right)$$

[5] We drop the subscript i for simplicity of notation.

But $q(\theta_1) - q(\theta_2) = q(\theta_3) - q(\theta_4) \neq 0$ *and* $\dot{u} \neq \ddot{u}$*; therefore (1) does not equal (2), in contradiction to (3). Now, if it is the case that* $\frac{\partial q(\theta)/\partial\theta^1}{\partial q(\theta)/\partial\theta^2} \neq \frac{\partial u(\theta)/\partial\theta^1}{\partial u(\theta)/\partial\theta^2}$ *for some* θ *then the indifference curves of* $u(\theta^1, \theta^2)$ *and* $q(\theta^1, \theta^2)$ *do not coincide but do intersect and therefore four such vectors exist. This implies that* $q(\theta)$ *is not Bayesian implementable. Figure 1 shows us these points. Two indifference curves of* $u(\theta^1, \theta^2)$*,* U_0 *and* U_1*, both intersect with two indifference curves of* $q(\theta^1, \theta^2)$*,* Q_0 *and* Q_1*. The intersection points satisfy the above conditions.*

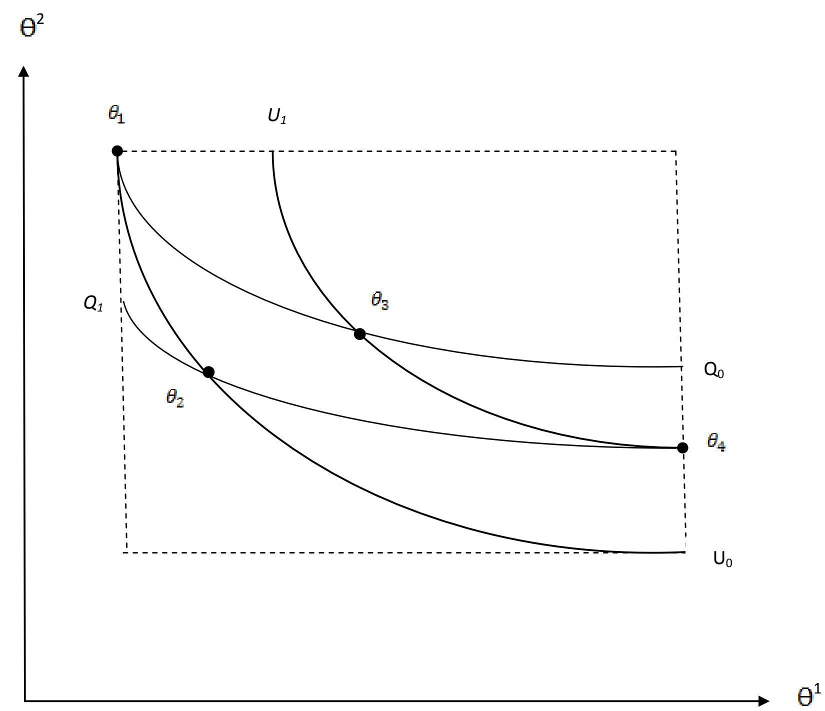

Figure 1: The Single Alternative Case

The results of this section show severe limitations on standard implementation of decision rules from the second subset. We see that the decision rules that the designer can implement by standard mechanisms are compelled to assign outcomes according to valuations. Is it the end of the road for the designer, or can she somehow escape this verdict. We discuss this in the following section.

5 Implementation by Extended Mechanisms

In the previous section we saw that the property of standard implementation basically restricts the decision rule's domain to the set of possible valuations. This raises the question what other measures can the designer take in order to allow for the implementation of decision rules that do not have this property. The measures that the designer can take depend on the economic environment in which she operates. For example, if the economic environment allows the designer to assign personal transfers to the agents, then the designer can apply standard mechanisms. We consider environments in which the designer has, in addition to the ability to assign personal transfers, the ability to affect the values of the arguments that compose the agents' valuation functions. This would enable the designer to apply more elaborate mechanisms. The example in Section 2 describes such an environment. The board of directors can change, by assigning transfers, the profits of the firm's departments, and the profits of two of the firm's department are the arguments of the valuation function of the CEO.

The ways in which the designer can affect each argument are characterized by a set of real-valued functions. We denote by F_i^j, $j \in \{1,2\}$ the set of functions that the designer can apply on j's signal of agent i:

$$F_i^j = \{f : \mathbb{R} \rightarrow \mathbb{R} \,|\, \textit{the designer can apply } f \textit{ on } j's \textit{ signal of agent } i\ \}$$

We assume that for every $i \in I$ and $j \in \{1,2\}$, the identity function is in F_i^j; i.e., the designer can choose not to apply the instruments that affect the arguments of the agents' valuation functions.

We characterize the outcomes that are achievable by a designer in such environments through the concept of implementation of extended social choice functions. This concept characterizes the equilibrium outcomes the designer can achieve by applying mechanisms that result in (1) a decision on an alternative (2) applying functions on the arguments of the agents valuation functions (3) assigning personal transfers.[6]

Extended social choice functions represent such direct mechanisms. An *extended*

[6]It characterizes the outcomes of truthful equilibria in direct mechanisms and due to the revelation principle it also characterizes the outcomes in general mechanisms.

social choice function is a function

$$S(\theta) = \left(q(\theta), \left(f_i^1(\theta, \cdot), f_i^2(\theta, \cdot), t_i(\theta)\right)_{i \in I}\right)$$

where $q : \Theta \to A$, $f_i^1 : \Theta \to F_i^1$, $f_i^2 : \Theta \to F_i^2$ and $t_i : \Theta \to \mathbb{R}$.

An extended social choice function is said to be *dominant-strategy implementable* if truthtelling is a dominant-strategy for every agent in the direct mechanism that is induced by it. Namely, for every $i \in I$ and every $\theta \in \Theta$ we have

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} u_i^{q\left(\hat{\theta}_i, \theta_{-i}\right)} \left(f_i^1\left(\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i^{q\left(\hat{\theta}_i, \theta_{-i}\right), 1}\right), f_i^2\left(\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i^{q\left(\hat{\theta}_i, \theta_{-i}\right), 2}\right)\right) + t_i\left(\hat{\theta}_i, \theta_{-i}\right)$$

A decision rule is *dominant-strategy implementable by an extended mechanism* if there exists a dominant-strategy implementable extended social choice function that includes it. An extended social choice function is *Bayesian implementable* if there exists a truthful Bayes–Nash equilibrium in the direct mechanism that is induced by it. A decision rule is *Bayesian implementable by an extended mechanism* if there exists a Bayesian implementable extended social choice function that includes it.[7]

We provide conditions for an economic setup and a decision rule that are sufficient for the extended implementability of the decision rule in the given setup. We use these conditions to derive the implementation of decision rules that assign different outcomes to profiles of signals that result in the same profile of valuations. Such decision rules, as we showed previously, are not implementable by standard mechanisms. The implementation of such decision rules by extended mechanisms becomes possible because in an extended mechanism inducing truthtelling does not impose the same constraints on the transfer functions that are imposed on them in a standard mechanism. These constraints prevent the implementability of these decision rules in the standard case.

The construction of the conditions is based on the insight that if there is a profile of valuation functions such that the decision rule assigns the same outcomes to profiles of signals that result in the same profile of valuations, then given these valuation functions it is possible to consider implementation by standard mechanisms. Our aim is to use the tools provided by the extended mechanism to modify agents'

[7]See the Appendix for a more formal exposition of these definitions.

original valuation functions to this desirable profile of valuation functions. Then, if standard implementability is possible given these desirable valuation functions, we can infer extended implementability in the setup of the original valuation functions.

The sufficient conditions are:

(1) The decision rule is implementable by a standard mechanism for some profile of valuation functions, call them *designer-preferred valuation functions.*

(2) We can modify agents valuation functions to coincide with designer-preferred valuation functions by the following procedure:

- For each $a \in A$ define Θ_i^a as the set of reports.
- Apply the incentive tools such that, given that alternative a is chosen, the agent will choose to report truthfully.
- Given a truthful report, the utility coincides with the designer-preferred valuation function for alternative a.

We now present this argument formally. We call a setup with the properties described in Section 3 a *U-setup*. Given the signal space of the U-setup, Θ, and the designer-preferred valuation functions, $\left(w_i^a\left(\theta_i\right)\right)_{a\in A, i\in I}$, we define a new setup, with an |A|-dimensional signal space for each agent, in which agents' signals coincide with the designer-preferred valuations. We call it a *W-setup*, and construct it as follows.

We define a function that maps each of agent i's original signals to the vector of the designer-preferred valuation, $W_i\left(\theta_i\right) = \left(w_i^a\left(\theta_i\right)\right)_{a\in A}$. We define agent i's signal space to be the image of this function, $W_i\left(\Theta_i\right)$, and denote it by W_i. Given that alternative a is chosen, her signal $w_i \in W_i$ and a transfer t_i agent i's utility is, $w_i^a + t_i$. The probability measure on W_i, η_i is derived from the U-setup probability measure, μ_i. Namely, the measure for event $B \subseteq W_i$ is $\eta_i\left(B\right) = \mu_i\left(C\left(B\right)\right)$, where $C\left(B\right) = \left\{L \subseteq \Theta_i \ s.t\ W_i\left(L\right) = B\right\}$.

We now present the modification procedure of the original valuation functions to the designer-preferred valuation functions. Given a U-setup and a W-setup we define:

- A W-setup is said to be *derived from a U-setup* if there exist functions $f_i^{a,1} : \Theta_i^a \to F_i^1$, $f_i^{a,2} : \Theta_i^a \to F_i^2$ and $t_i^a : \Theta_i^a \to \mathbb{R}\ \forall a \in A\ \forall i \in I$ such that for every $i \in I$, every $a \in A$, and every $\theta_i^a \in \Theta_i^a$ we have

$$(1) \qquad \theta_i^a \in \arg\max_{\hat{\theta}_i^a \in \Theta_i^a} u_i^a \left(f_i^{a,1} \left(\hat{\theta}_i^a, \theta_i^{a,1} \right), f_i^{a,2} \left(\hat{\theta}_i^a, \theta_i^{a,2} \right) \right) + t_i^a \left(\hat{\theta}_i^a \right)$$

and in addition

$$(2) \qquad u_i^a \left(f_i^{a,1} \left(\theta_i^a, \theta_i^{a,1} \right), f_i^{a,2} \left(\theta_i^a, \theta_i^{a,2} \right) \right) + t_i^a \left(\theta_i^a \right) = w_i^a \left(\theta_i^{a,1}, \theta_i^{a,2} \right)$$

Theorem 7. *Assume a U-setup. And assume a W-setup that is derived from this U-setup. If $q(w)$ is a decision rule that is dominant-strategy implementable (Bayesian implementable) by a standard mechanism in the W-setup, then the decision rule $q(\theta) = q(w(\theta))$ is dominant-strategy implementable (Bayesian implementable) by an extended mechanism in the U-setup.*

The structure of the proof is as follows.[8] Given a profile of transfers $(\tau_i(w))_{i \in I}$ that implements $q(w)$ in the W-setup we define the following functions $f_i^1(\theta) = f_i^{q(\theta),1} \left(\theta_i^{q(\theta)}, \cdot \right)$, $f_i^2(\theta) = f_i^{q(\theta),2} \left(\theta_i^{q(\theta)}, \cdot \right)$ and $t_i(\theta) = t_i^{q(\theta)} \left(\theta_i^{q(\theta)} \right) + \tau_i(w_i(\theta_i), w_{-i}(\theta_{-i}))$. Consider some agent i and a profile of signals of the agents other than i, θ_{-i}. Assume that alternative a is chosen when agent i reports truthfully, i.e., $q(\theta) = a$. Assume agent i reports an untrue signal, θ_i', such that $q\left(\theta_i', \theta_{-i}\right) = a$. The assumption that $q(w)$ is dominant-strategy implementable by the transfer scheme $(\tau_i(w))_{i \in I}$ implies that $\tau_i\left(\theta_i', \theta_{-i}\right) = \tau_i(\theta)$. Condition (1) then implies that agent i does not gain from this deviation. Assume that agent i reports an untrue signal, $\tilde{\theta}_i$ such that $q\left(\tilde{\theta}_i, \theta_{-i}\right) = b$ and $\left(\tilde{\theta}_i^{b,1}, \tilde{\theta}_i^{b,2}\right) = \left(\theta_i^{b,1}, \theta_i^{b,2}\right)$. The assumption that $q(w)$ is dominant-strategy implementable by the transfer scheme $(\tau_i(w))_{i \in I}$ implies that agent i does not gain from this deviation. Condition (1) then implies that agent i does not gain if she reports any untrue signal $\hat{\theta}_i$ such that $q\left(\hat{\theta}_i, \theta_{-i}\right) = b$.

Consider a U-setup. Theorem 7 shows that we can implement by an extended mechanism any decision rule that can be implemented by a standard mechanism in some W-setup, provided that this W-setup can be derived from the U-setup. This result relates to the existing literature on implementation in independent private values

[8] The proof appears in the Appendix.

models, in which the signal space is $|A|$-dimensional and the agents' signals are their valuation profiles.[9] Once we have derived a W-setup from the U-setup we can take these results, which allow us to infer implementability by a standard mechanism in the W-setup, and deduce implementability by an extended mechanism in the U-setup. For example, we can implement the efficient decision rule of the W-setup.

5.1 Derived Valuations

In this subsection and in the Appendix we present examples of designer-preferred valuation functions that can be derived from various common forms of multivariate valuation functions. These modifications allow us to implement, by extended mechanisms, decision rules that assign outcomes according to these designer-preferred valuation functions. Since these designer-preferred valuation functions do not coincide with the agents' original valuation functions, we can infer the extended implementatability of decision rules that do not assign the same outcome to profiles of signals that result in the same profile of valuations.

We consider the most natural case in which the designer can affect an argument's value by assigning a transfer to it. Namely, the sets F_i^j are defined as follows:

$$F_i^j = \{f : \mathbb{R} \to \mathbb{R} \mid s.t\ f(x) = x + c\ ,\ c \in \mathbb{R}\}$$

Given these F_i^js, a signal $\theta_i^{a,1}$ and report $\hat{\theta}_i^a$ an agent's utility is

$$u_i^a\left(\theta_i^{a,1} + t_i^{a,1}\left(\hat{\theta}_i^a\right), \theta_i^{a,2} + t_i^{a,2}\left(\hat{\theta}_i^a\right)\right) + t_i^{a,3}\left(\hat{\theta}_i^a\right)$$

5.1.1 Cobb–Douglas Valuation Functions

Assume that the valuation function for alternative a *takes the following form:*

$$u_i^a\left(\theta_i^{a,1}, \theta_i^{a,2}\right) = \left(\theta_i^{a,1}\right)^{\alpha}\left(\theta_i^{a,2}\right)^{1-\alpha} \qquad 0 < \alpha < 1$$

[9] See, e.g, Groves (1973), Myerson (1981), Roberts (1979), Rochet (1987), Bikhchandani et al. (2006).

Define, $t_i^{a,1}\left(\hat{\theta}_i^a\right)=\hat{\theta}_i^{a,2}$, $t_i^{a,2}\left(\hat{\theta}_i^a\right)=\hat{\theta}_i^{a,1}$ *and* $t_i^{a,3}\left(\hat{\theta}_i^a\right)=-(1-\alpha)\hat{\theta}_i^{a,1}-\alpha\hat{\theta}_i^{a,2}$. *Given signal* θ_i^a *and report* $\hat{\theta}_i^a$ *, the utility is*

$$\left(\theta_i^{a,1}+\hat{\theta}_i^{a,2}\right)^{\alpha}\left(\theta_i^{a,2}+\hat{\theta}_i^{a,1}\right)^{1-\alpha}-(1-\alpha)\hat{\theta}_i^{a,1}-\alpha\hat{\theta}_i^{a,2}$$

For each signal, truthtelling is an optimal action:

$$\theta_i^a\in\arg\max_{\hat{\theta}_i^a\in\Theta_i^a}\left(\theta_i^{a,1}+\hat{\theta}_i^{a,2}\right)^{\alpha}\left(\theta_i^{a,2}+\hat{\theta}_i^{a,1}\right)^{1-\alpha}-(1-\alpha)\hat{\theta}_i^{a,1}-\alpha\hat{\theta}_i^{a,2}$$

Given a truthful report , the value of the utility is

$$\alpha\theta_i^{a,1}+(1-\alpha)\,\theta_i^{a,2}$$

We get that the weighted mean of the signals with parameter α *can be derived from a Cobb–Douglas valuation that has the same parameter.*

6 Concluding Remarks

We have considered the possibility of implementation in models of independent private values in which an agent's valuation of an alternative is composed of multiple arguments. We have shown that when we apply standard mechanisms, only decision rules whose domain is the set of possible valuations are eligible for implementation. However, when we consider extended mechanisms this restriction is not necessary for implementation. We did not, however, provide a simple and tractable characterization of the set of all the decision rules that are implemented by extended mechanisms. Such a characterization will be the subject of future work.

A Appendix

A.1 Examples of Derived Valuations

A.1.1 Product Valuation Functions

Assume that the valuation function for alternative a *takes the following form:*

$$u_i^a\left(\theta_i^{a,1},\theta_i^{a,2}\right)=\theta_i^{a,1}\cdot\theta_i^{a,2}$$

Define $t_i^{a,1}\left(\hat{\theta}_i^a\right)=\hat{\theta}_i^{a,2}$, $t_i^{a,2}\left(\hat{\theta}_i^a\right)=\hat{\theta}_i^{a,1}$, *and* $t_i^{a,3}\left(\hat{\theta}_i^a\right)=-\frac{\left(\hat{\theta}_i^{a,1}+\hat{\theta}_i^{a,2}\right)^2}{2}$.
Given signal θ_i^a *and report* $\hat{\theta}_i^a$ *, the utility* is

$$\left(\theta_i^{a,1}+\hat{\theta}_i^{a,2}\right)\left(\theta_i^{a,2}+\hat{\theta}_i^{a,1}\right)-\frac{\left(\hat{\theta}_i^{a,1}+\hat{\theta}_i^{a,2}\right)^2}{2}$$

For each signal, truthtelling is an optimal action:

$$\theta_i^a\in\arg\max_{\hat{\theta}_i^a\in\Theta_i^a}\left(\theta_i^{a,1}+\hat{\theta}_i^{a,2}\right)\left(\theta_i^{a,2}+\hat{\theta}_i^{a,1}\right)-\frac{\left(\hat{\theta}_i^{a,1}+\hat{\theta}_i^{a,2}\right)^2}{2}$$

Given a truthful report , the value of the utility is

$$\frac{\left(\theta_i^{a,1}+\theta_i^{a,2}\right)^2}{2}$$

The square of the sum of signals can be derived from a product valuation.

A.1.2 Perfect Substitutes Valuation Functions

Assume that the valuation function for alternative a *takes the following form:*

$$u_i^a\left(\theta_i^{a,1},\theta_i^{a,2}\right)=\alpha\cdot\theta_i^{a,1}+\theta_i^{a,2}$$

To ensure that a report is truthful given any signal θ_i^a*, implies that the transfer functions must add up to a constant. Therefore, the only valuation that can be derived from a* *perfect substitute valuation is equal to the original valuation plus a constant.*

A.1.3 Quasilinear Valuation Functions

Assume that the valuation function for alternative a *takes the following form:*

$$u_i^a\left(\theta_i^{a,1},\theta_i^{a,2}\right)=f\left(\theta_i^{a,1}\right)+\theta_i^{a,2}$$

Assume $m>1$ and assume that $f''\left(m\cdot\theta_i^{a,1}\right)>0$ for every $\theta_i^{a,1}$.
Define, $t_i^{a,1}\left(\hat{\theta}_i^a\right)=(m-1)\,\hat{\theta}_i^{a,1}$, $t_i^{a,2}\left(\hat{\theta}_i^a\right)=-\frac{m-1}{m}f\left(m\cdot\hat{\theta}_i^{a,1}\right)$, *and* $t_i^{a,3}\left(\hat{\theta}_i^a\right)=0$.
Given signal θ_i^a and report $\hat{\theta}_i^a$, the utility is

$$f\left(\theta_i^{a,1}+(m-1)\,\hat{\theta}_i^{a,1}\right)-\left(\frac{m-1}{m}\right)f\left(m\cdot\hat{\theta}_i^{a,1}\right)+\theta_i^{a,2}$$

For each signal, truthtelling is an optimal action:

$$\theta_i^a\in\underset{\hat{\theta}_i^a\in\Theta_i^a}{\arg\max}\,f\left(\theta_i^{a,1}+(m-1)\,\hat{\theta}_i^{a,1}\right)-\left(\frac{m-1}{m}\right)f\left(m\cdot\hat{\theta}_i^{a,1}\right)+\theta_i^{a,2}$$

Given a truthful report , the value of the utility is

$$\frac{1}{m}\cdot f\left(m\cdot\theta_i^{a,1}\right)+\theta_i^{a,2}$$

We get that we can derive a quasilinear valuation function in which the non linear function is the average value of the original non linear function, f, as a function of m, when we fix $\theta_i^{a,1}$ as a parameter.

A.2 Definitions

- A decision rule $q(\theta)$ is *implementable in dominant strategies by an extended mechanism* if there exists a profile of functions $\left(f_i^1(\theta,\cdot),f_i^2(\theta,\cdot),t_i(\theta)\right)_{i\in I}$ such that

$$\left(q(\theta),\left(f_i^1(\theta,\cdot),f_i^2(\theta,\cdot),t_i(\theta)\right)_{i\in I}\right)$$

is implementable in dominant strategies.

- An extended social choice function is said to be *Bayesian implementable* if for every $i \in I$ and every $\theta_i \in \Theta_i$ we have

$$\theta_i \in \arg\max_{\hat{\theta}_i \in \Theta_i} E_{\theta_{-i}} \left[u_i^{q\left(\hat{\theta}_i, \theta_{-i}\right)} \left(f_i^1 \left(\left(\hat{\theta}_i, \theta_{-i} \right), \theta_i^{q\left(\hat{\theta}_i, \theta_{-i}\right),1} \right), f_i^2 \left(\left(\hat{\theta}_i, \theta_{-i} \right), \theta_i^{q\left(\hat{\theta}_i, \theta_{-i}\right),2} \right) \right) + t_i \left(\hat{\theta}_i, \theta_{-i} \right) \right]$$

- A decision rule $q(\theta)$ is *Bayesian implementable by an extended mechanism* if there exists a profile of functions $\left(f_i^1(\theta, \cdot), f_i^2(\theta, \cdot), t_i(\theta)\right)_{i \in I}$ such that

$$\left(q(\theta), \left(f_i^1(\theta, \cdot), f_i^2(\theta, \cdot), t_i(\theta) \right)_{i \in I} \right)$$

is Bayesian implementable.

A.3 Proofs

A.3.1 Proof of Theorem 4

Consider some $\theta_i^{a,1}$; we start by developing the following partial derivative $\frac{\partial \tau_i(\theta_i)}{\partial \theta_i^{a,1}}$.
We denote $\theta_i(h) = \left(\theta_i^{1,1}, \theta_i^{1,2}, \theta_i^{2,1}, ..., \theta_i^{a,1} + h, \theta_i^{a,2}, ..., \theta_i^{|A|,2}\right)$ for every h. We now develop the following limit:

$$\lim_{h \to 0} \frac{\tau_i(\theta_i(h)) - \tau_i(\theta_i)}{h}$$

Bayesian implementability implies that

$$V_i(\theta_i) \cdot q_i(\theta_i) + \tau_i(\theta_i) \geq V_i(\theta_i) \cdot q_i(\theta_i(h)) + \tau_i(\theta_i(h))$$

and therefore

$$V_i(\theta_i) \cdot (q_i(\theta_i) - q_i(\theta_i(h))) \geq \tau_i(\theta_i(h)) - \tau_i(\theta_i)$$

It also implies that

$$V_i(\theta_i(h)) \cdot q_i(\theta_i(h)) + \tau_i(\theta_i(h)) \geq V_i(\theta_i(h)) \cdot q_i(\theta_i) + \tau_i(\theta_i)$$

and therefore

$$\tau_i(\theta_i(h)) - \tau_i(\theta_i) \geq V_i(\theta_i(h)) \cdot (q_i(\theta_i) - q_i(\theta_i(h)))$$

We conclude that

$$V_i(\theta_i) \cdot (q_i(\theta_i) - q_i(\theta_i(h))) \geq \tau_i(\theta_i(h)) - \tau_i(\theta_i) \geq V_i(\theta_i(h)) \cdot (q_i(\theta_i) - q_i(\theta_i(h)))$$

Dividing all the elements in the inequality by $h > 0$ and looking at the limit $h \to 0_+$, we get

$$\left[\sum_{b \in A \setminus \{a\}} u_i^b\left(\theta_i^{b,1}, \theta_i^{b,2}\right) \frac{\left(q_i^b(\theta_i) - q_i^b(\theta_i(h))\right)}{h}\right] + u_i^a\left(\theta_i^{a,1}, \theta_i^{a,2}\right) \frac{(q_i^a(\theta_i) - q_i^a(\theta_i(h)))}{h} \geq$$

$$\geq \frac{\tau_i(\theta_i(h)) - \tau_i(\theta_i)}{h} \geq$$

$$\geq \left[\sum_{b \in A \setminus \{a\}} u_i^b\left(\theta_i^{b,1}, \theta_i^{b,2}\right) \frac{\left(q_i^b(\theta_i) - q_i^b(\theta_i(h))\right)}{h}\right] + u_i^a\left(\theta_i^{a,1} + h, \theta_i^{a,2}\right) \frac{(q_i^a(\theta_i) - q_i^a(\theta_i(h)))}{h}$$

Now the limit of the LHS is

$$\lim_{h \to 0_+} \left[\sum_{b \in A \setminus \{a\}} u_i^b\left(\theta_i^{b,1}, \theta_i^{b,2}\right) \frac{\left(q_i^b(\theta_i) - q_i^b(\theta_i(h))\right)}{h}\right] + u_i^a\left(\theta_i^{a,1}, \theta_i^{a,2}\right) \frac{(q_i^a(\theta_i) - q_i^a(\theta_i(h)))}{h} =$$

$$= \sum_{c \in A} u_i^c\left(\theta_i^{c,1}, \theta_i^{c,2}\right)\left(-\frac{\partial q_i^c(\theta_i)}{\partial \theta_i^{a,1}}\right)$$

and the limit of the RHS is

$$\lim_{h \to 0_+} \left[\sum_{b \in A \setminus \{a\}} u_i^b\left(\theta_i^{b,1}, \theta_i^{b,2}\right) \frac{\left(q_i^b(\theta_i) - q_i^b(\theta_i(h))\right)}{h}\right] + u_i^a\left(\theta_i^{a,1} + h, \theta_i^{a,2}\right) \frac{(q_i^a(\theta_i) - q_i^a(\theta_i(h)))}{h} =$$

$$= \sum_{c \in A} u_i^c\left(\theta_i^{c,1}, \theta_i^{c,2}\right)\left(-\frac{\partial q_i^c(\theta_i)}{\partial \theta_i^{a,1}}\right)$$

We get from the sandwich theorem that

$$\lim_{h \to 0_+} \frac{\tau_i(\theta_i(h)) - \tau_i(\theta_i)}{h} = \sum_{c \in A} u_i^c\left(\theta_i^{c,1}, \theta_i^{c,2}\right)\left(-\frac{\partial q_i^c(\theta_i)}{\partial \theta_i^{c,1}}\right)$$

Similarly, dividing all the elements in the inequality by $h < 0$ and looking at the limit $h \to 0_-$, we get

$$\lim_{h\to 0_-} \frac{\tau_i\left(\theta_i(h)\right) - \tau_i\left(\theta_i\right)}{h} = \sum_{c\in A} u_i^c\left(\theta_i^{c,1}, \theta_i^{c,2}\right)\left(-\frac{\partial q_i^c\left(\theta_i\right)}{\partial \theta_i^{c,1}}\right)$$

We conclude that,

$$\frac{\partial \tau_i\left(\theta_i\right)}{\partial \theta_i^{a,1}} = \sum_{c\in A} u_i^c\left(\theta_i^{c,1}, \theta_i^{c,2}\right)\left(-\frac{\partial q_i^c\left(\theta_i\right)}{\partial \theta_i^{a,1}}\right)$$

We define $U\left(\theta_i\right)$ to be the expected utility of agent i in a truthtelling Bayesian equilibrium:

$$U_i(\theta_i) = V_i\left(\theta_i\right)\cdot q_i(\theta_i) + t_i\left(\theta_i\right) = \sum_{b\in A} u_i^b\left(\theta_i^{b,1}, \theta_i^{b,2}\right) q_i^b\left(\theta_i\right) + \tau_i\left(\theta_i\right)$$

Differentiating $U_i(\theta_i)$ with respect to $\theta_i^{a,1}$, we get[10]

$$\frac{\partial U_i}{\partial \theta_i^{a,1}}\left(\theta_i\right) = \sum_{b\in A} \frac{\partial u_i^b}{\partial \theta_i^{a,1}}\left(\theta_i\right)\cdot q_i^b\left(\theta_i\right) + \sum_{b\in A} u_i^b\left(\theta_i^{b,1}, \theta_i^{b,2}\right)\frac{\partial q_i^b\left(\theta_i\right)}{\partial \theta_i^{a,1}} + \frac{\partial \tau_i\left(\theta_i\right)}{\partial \theta_i^{a,1}} =$$

$$= \frac{\partial u_i^a}{\partial \theta_i^{a,1}}\left(\theta_i\right) q_i^a\left(\theta_i\right)$$

and differentiating with respect to $\theta_i^{a,2}$, we get

$$\frac{\partial^2 U_i}{\partial \theta_i^{a,2}, \partial \theta_i^{a,1}}\left(\theta_i\right) = \frac{\partial^2 u_i^a}{\partial \theta_i^{a,2}, \partial \theta_i^{a,1}}\left(\theta_i\right) q_i^a\left(\theta_i\right) + \frac{\partial q_i^a}{\partial \theta_i^{a,2}}\left(\theta_i\right)\frac{\partial u_i^a}{\partial \theta_i^{a,1}}\left(\theta_i\right)$$

Differentiating $U\left(\theta_i\right)$ with respect to $\theta_i^{a,2}$ and then with respect to $\theta_i^{a,1}$, we get

$$\frac{\partial^2 U_i}{\partial \theta_i^{a,1}, \partial \theta_i^{a,2}}\left(\theta_i\right) = \frac{\partial^2 u_i^a}{\partial \theta_i^{a,1}, \partial \theta_i^{a,2}}\left(\theta_i\right) q_i^a\left(\theta_i\right) + \frac{\partial q_i^a}{\partial \theta_i^{a,1}}\left(\theta_i\right)\frac{\partial u_i^a}{\partial \theta_i^{a,2}}\left(\theta_i\right)$$

Since u_i^a is twice continuously differentiable and by Schwartz's theorem, we get

$$\frac{\partial^2 u_i^a}{\partial \theta_i^{a,1}, \partial \theta_i^{a,2}}\left(\theta_i\right) = \frac{\partial^2 u_i^a}{\partial \theta_i^{a,2}, \partial \theta_i^{a,1}}\left(\theta_i\right)$$

and, in addition, since $q_i\left(\theta_i\right)$ is continuously differentiable and by the Schwartz's theorem, we

[10]We could apply the envelope theorem, but since the partial derivatives of τ_i have already been developed, we choose a straight-forward approach.

get

$$\frac{\partial^2 U_i}{\partial \theta_i^{a,2}, \partial \theta_i^{a,1}} (\theta_i) = \frac{\partial^2 U_i}{\partial \theta_i^{a,1}, \partial \theta_i^{a,2}} (\theta_i)$$

and so

$$\frac{\partial q_i^a}{\partial \theta_i^{a,2}} (\theta_i) \frac{\partial u_i^a}{\partial \theta_i^{a,1}} (\theta_i) = \frac{\partial q_i^a}{\partial \theta_i^{a,1}} (\theta_i) \frac{\partial u_i^a}{\partial \theta_i^{a,2}} (\theta_i)$$

Since $\frac{\partial u_i^a}{\partial \theta_i^{a,1}} (\theta_i) \neq 0$ and $\frac{\partial u_i^a}{\partial \theta_i^{a,2}} (\theta_i) \neq 0$, it follows that for every $\theta_i \in \Theta_i$, $\frac{\partial q_i^a}{\partial \theta_i^{a,1}} (\theta_i) = 0$ iff $\frac{\partial q_i^a}{\partial \theta_i^{a,2}} (\theta_i) = 0$ and otherwise

$$\frac{\partial q_i^a(\theta_i)/\partial \theta_i^{a,1}}{\partial q_i^a(\theta_i)/\partial \theta_i^{a,2}} = \frac{\partial u_i^a(\theta_i)/\partial \theta_i^{a,1}}{\partial u_i^a(\theta_i)/\partial \theta_i^{a,2}}$$

We now show the equality for $b \in A \setminus \{a\}$.
Differentiating $U_i(\theta_i)$ with respect to $\theta_i^{b,1}$, we get

$$\frac{\partial U_i}{\partial \theta_i^{b,1}} (\theta_i) = \frac{\partial u_i^b}{\partial \theta_i^{b,1}} (\theta_i) \, q_i^b (\theta_i)$$

and differentiating with respect to $\theta_i^{a,1}$, we get

$$\frac{\partial^2 U_i}{\partial \theta_i^{a,1}, \partial \theta_i^{b,1}} (\theta_i) = \frac{\partial u_i^b}{\partial \theta_i^{b,1}} (\theta_i) \frac{\partial q_i^b}{\partial \theta_i^{a,1}} (\theta_i)$$

Differentiating $U(\theta_i)$ with respect to $\theta_i^{a,1}$ and then with respect to $\theta_i^{b,1}$, we get

$$\frac{\partial^2 U_i}{\partial \theta_i^{b,1}, \partial \theta_i^{a,1}} (\theta_i) = \frac{\partial u_i^a}{\partial \theta_i^{a,1}} (\theta_i) \frac{\partial q_i^a}{\partial \theta_i^{b,1}} (\theta_i)$$

Since u_i^a , u_i^b , q_i^a and q_i^b are continuously differentiable and by Schwartz's theorem, we get

$$\frac{\partial^2 U_i}{\partial \theta_i^{a,1}, \partial \theta_i^{b,1}} (\theta_i) = \frac{\partial^2 U_i}{\partial \theta_i^{b,1}, \partial \theta_i^{a,1}} (\theta_i)$$

and so

$$\frac{\partial q_i^a}{\partial \theta_i^{b,1}} (\theta_i) = \frac{\frac{\partial u_i^b}{\partial \theta_i^{b,1}} (\theta_i) \frac{\partial q_i^b}{\partial \theta_i^{a,1}} (\theta_i)}{\frac{\partial u_i^a}{\partial \theta_i^{a,1}} (\theta_i)}$$

Replacing $\theta_i^{a,1}$ with $\theta_i^{a,2}$ in the above derivation we get

$$\frac{\partial q_i^a}{\partial \theta_i^{b,1}}(\theta_i) = \frac{\frac{\partial u_i^b}{\partial \theta_i^{b,1}}(\theta_i)\frac{\partial q_i^b}{\partial \theta_i^{a,2}}(\theta_i)}{\frac{\partial u_i^a}{\partial \theta_i^{a,2}}(\theta_i)}$$

so

$$\frac{\frac{\partial q_i^b}{\partial \theta_i^{a,2}}(\theta_i)}{\frac{\partial u_i^a}{\partial \theta_i^{a,2}}(\theta_i)} = \frac{\frac{\partial q_i^b}{\partial \theta_i^{a,1}}(\theta_i)}{\frac{\partial u_i^a}{\partial \theta_i^{a,1}}(\theta_i)}$$

and we get $\frac{\partial q_i^b}{\partial \theta_i^{a,2}}(\theta_i) = 0$ iff $\frac{\partial q_i^b}{\partial \theta_i^{a,1}}(\theta_i) = 0$ and otherwise

$$\frac{\partial q_i^b(\theta_i)/\partial \theta_i^{a,1}}{\partial q_i^b(\theta_i)/\partial \theta_i^{a,2}} = \frac{\partial u_i^a(\theta_i)/\partial \theta_i^{a,1}}{\partial u_i^a(\theta_i)/\partial \theta_i^{a,2}}$$

which is what we wanted to prove □

A.3.2 Corollary

Assume that $q(\theta)$ is Bayesian implementable and that for every $i \in I$, $q_i(\theta_i)$ is continuously differentiable. Let $\theta_i = (\theta_i^a)_{a\in A}$ and $\tilde{\theta}_i = \left(\tilde{\theta}_i^a\right)_{a\in A}$ be signals such that $u_i^a(\theta_i^a) = u_i^a\left(\tilde{\theta}_i^a\right)$ for every $a \in A$ then $q_i(\theta_i) = q_i\left(\tilde{\theta}_i\right)$

Proof

Look at $q_i\left(\tilde{\theta}_i^a, (\theta_i^c)_{c\in A\setminus\{a\}}\right)$ by Theorem 4 it equals to $q_i(\theta_i)$. Now look at $q_i\left(\tilde{\theta}_i^a, \tilde{\theta}_i^b, (\theta_i^c)_{c\in A\setminus\{a,b\}}\right)$ by Theorem 4 it equals to $q_i\left(\tilde{\theta}_i^a, (\theta_i^c)_{c\in A\setminus\{a\}}\right)$. Continue in the same way for the rest of the alternatives.

A.3.3 Proof of Theorem 7

Assume $q(w)$ is implementable in some W-setup and assume that the W-setup is derived from the U-setup by the functions $\left(f_i^{a,1}(\theta_i^a,\cdot),\ f_i^{a,2}(\theta_i^a,\cdot),\ t_i^a(\theta_i^a)\right)_{a\in A, i\in I}$. Define the following functions

$$f_i^1(\theta,\cdot) = f_i^{q(\theta),1}(\theta_i^{q(\theta)},\cdot)$$

$$f_i^2(\theta,\cdot) = f_i^{q(\theta),2}(\theta_i^{q(\theta)},\cdot)$$

$$t_i(\theta) = t_i^{q(\theta)}\left(\theta_i^{q(\theta)}\right) + \tau_i\left(w_i(\theta_i), w_{-i}(\theta_{-i})\right)$$

We start by proving the theorem for dominant-strategy implementation. Since $q(w)$ is implementable in dominant strategies in the W-setup by a standard mechanism, we have a standard SCF $(q(w), \tau_1(w), ..., \tau_n(w))$ such that for every $i \in I$ and every $w \in W$ we have

$$w_i^{q(w)} + \tau_i(w) \geq w_i^{q(\hat{w}_i, w_{-i})} + \tau_i(\hat{w}_i, w_{-i})$$

for every $\hat{w}_i \in W_i$.

Let $\theta_{-i} \in \Theta_{-i}$ and assume that $q(\theta) = q(w_i(\theta_i), w_{-i}(\theta_{-i})) = a$.

Assume agent i deviates from a truthful report and reports some $\hat{\theta}_i$ such that $q(\theta) = q\left(\hat{\theta}_i, \theta_{-i}\right) = a$; then $\tau_i(w_i(\theta_i), w_{-i}(\theta_{-i})) = \tau_i\left(w_i\left(\hat{\theta}_i\right), w_{-i}(\theta_{-i})\right) = \tau_i(a, w_{-i}(\theta_{-i}))$.

If agent i reports truthfully, her utility is

$$u_i^a\left(f_i^{a,1}\left(\theta_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\theta_i^a, \theta_i^{a,2}\right)\right) + t_i^a(\theta_i^a) + \tau_i(w_i(\theta_i), w_{-i}(\theta_{-i}))$$

and her utility if she reports $\hat{\theta}_i$ is

$$u_i^a\left(f_i^{a,1}\left(\hat{\theta}_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\hat{\theta}_i^a, \theta_i^{a,2}\right)\right) + t_i^{a,3}\left(\hat{\theta}_i^a\right) + \tau_i\left(w_i\left(\hat{\theta}_i\right), w_{-i}(\theta_{-i})\right)$$

However, since

$$\theta_i^a \in \underset{\hat{\theta}_i^a \in \Theta_i^a}{\arg\max}\, u_i^a\left(f_i^{a,1}\left(\hat{\theta}_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\hat{\theta}_i^a, \theta_i^{a,2}\right)\right) + t_i^{a,3}\left(\hat{\theta}_i^a\right)$$

and

$$\tau_i(w_i(\theta_i), w_{-i}(\theta_{-i})) = \tau_i\left(w_i\left(\hat{\theta}_i\right), w_{-i}(\theta_{-i})\right) = \tau_i(a, w_{-i}(\theta_{-i}))$$

such deviation is not profitable.

Assume agent i deviates from a truthful report and reports some $\hat{\theta}_i$ such that $q\left(\hat{\theta}_i, \theta_{-i}\right) = b$:

If agent i reports truthfully, her utility is

$$u_i^a\left(f_i^{a,1}(\theta_i^a, \theta_i^{a,1}), f_i^{a,2}(\theta_i^a, \theta_i^{a,2})\right) + t_i^a(\theta_i^a) + \tau_i(w_i(\theta_i), w_{-i}(\theta_{-i})) = w_i^a(\theta_i) + \tau_i(a, w_{-i}(\theta_{-i}))$$

Since $q(w)$ is dominant-strategy implementable by a standard mechanism in the W-setup this expression is greater than or equal to,

$$w_i^b(\theta_i) + \tau_i(b, w_{-i}(\theta_{-i}))$$

which is equal to

$$u_i^b\left(f_i^{b,1}(\theta_i^b, \theta_i^{b,1}), f_i^{b,2}(\theta_i^b, \theta_i^{b,2})\right) + t_i^b\left(\theta_i^b\right) + \tau_i(b, w_{-i}(\theta_{-i}))$$

which is greater than or equal to

$$u_i^b\left(f_i^{b,1}\left(\hat{\theta}_i^b, \theta_i^{b,1}\right), f_i^{b,2}\left(\hat{\theta}_i^b, \theta_i^{b,2}\right)\right) + t_i^b\left(\hat{\theta}_i^b\right) + \tau_i(b, w_{-i}(\theta_{-i}))$$

which is her utility if she reports $\hat{\theta}_i$. Therefore such deviation is not profitable.
We now prove the theorem for Bayesian implementation. Given a profile of $(\tau_1(w), ..., \tau_n(w))$, we denote $\tau_i(w_i) = E_{w_{-i}}[\tau_i(w)]$. Now $q(w)$ is Bayesian implementable in the W-setup by a standard mechanism, and so we have a SCF $(q(w), \tau_1(w), ..., \tau_n(w))$ such that for every $i \in I$ and every $w \in W$ we have,

$$\left[\sum_{a\in A} w_i^a \cdot q_i^a(w_i)\right] + \tau_i(w_i) \geq \left[\sum_{a\in A} w_i^a \cdot q_i^a(\hat{w}_i)\right] + \tau_i(\hat{w}_i)$$

for every $\hat{w}_i \in W_i$.
Given θ_i, assume that agent i deviates and reports a signal $\hat{\theta}_i$. Assume that $w_i(\theta_i) = w_i\left(\hat{\theta}_i\right)$; then $q_i(w_i(\theta_i)) = q_i\left(w_i\left(\hat{\theta}_i\right)\right)$ and $\tau_i(w_i(\theta_i)) = \tau_i\left(w_i\left(\hat{\theta}_i\right)\right)$.
Now agent i's expected utility if she reports truthfully is

$$\left[\sum_{a\in A}\left(u_i^a\left(f_i^{a,1}\left(\theta_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\theta_i^a, \theta_i^{a,2}\right)\right) + t_i^a(\theta_i^a)\right) \cdot q_i^a(w_i(\theta_i))\right] + \tau_i(w_i(\theta_i))$$

If she reports $\hat{\theta}_i$ her expected utility is

$$\left[\sum_{a\in A}\left(u_i^a\left(f_i^{a,1}\left(\hat{\theta}_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\hat{\theta}_i^a, \theta_i^{a,2}\right)\right) + t_i^{a,3}\left(\hat{\theta}_i^a\right)\right) \cdot q_i^a\left(w_i\left(\hat{\theta}_i\right)\right)\right] + \tau_i\left(w_i\left(\hat{\theta}_i\right)\right)$$

Now since $q_i(w_i(\theta_i)) = q_i\left(w_i\left(\hat{\theta}_i\right)\right)$ and $\tau_i(w_i(\theta_i)) = \tau_i\left(w_i\left(\hat{\theta}_i\right)\right)$ and since

$$\theta_i^a \in \arg\max_{\hat{\theta}_i^a \in \Theta_i^a} u_i^a\left(\theta_i^{a,1} + t_i^{a,1}\left(\hat{\theta}_i^a\right), \theta_i^{a,2} + t_i^{a,2}\left(\hat{\theta}_i^a\right)\right) + t_i^{a,3}\left(\hat{\theta}_i^a\right)$$

for every $a \in A$, her expected utility when she reports $\hat{\theta}_i$ is smaller or equal to her expected utility when she reports truthfully.

Assume that $w_i(\theta_i) \neq w_i\left(\hat{\theta}_i\right)$; then her expected utility is

$$\left[\sum_{a\in A}\left(u_i^a\left(f_i^{a,1}\left(\hat{\theta}_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\hat{\theta}_i^a, \theta_i^{a,2}\right)\right) + t_i^{a,3}\left(\hat{\theta}_i^a\right)\right) \cdot q_i^a\left(w_i\left(\hat{\theta}_i\right)\right)\right] + \tau_i\left(w_i\left(\hat{\theta}_i\right)\right)$$

which is less than or equal to

$$\left[\sum_{a\in A}\left(u_i^a\left(f_i^{a,1}\left(\theta_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\theta_i^a, \theta_i^{a,2}\right)\right) + t_i^a\left(\theta_i^a\right)\right) \cdot q_i^a\left(w_i\left(\hat{\theta}_i\right)\right)\right] + \tau_i\left(w_i\left(\hat{\theta}_i\right)\right)$$

which is equal to

$$\left[\sum_{a\in A} w_i^a(\theta_i) \cdot q_i^a\left(w_i\left(\hat{\theta}_i\right)\right)\right] + \tau_i\left(w_i\left(\hat{\theta}_i\right)\right)$$

which is less than or equal to

$$\left[\sum_{a\in A} w_i^a(\theta_i) \cdot q_i^a(w_i(\theta_i))\right] + \tau_i(w_i(\theta_i))$$

which is equal to

$$\left[\sum_{a\in A}\left(u_i^a\left(f_i^{a,1}\left(\theta_i^a, \theta_i^{a,1}\right), f_i^{a,2}\left(\theta_i^a, \theta_i^{a,2}\right)\right) + t_i^a\left(\theta_i^a\right)\right) \cdot q_i^a(w_i(\theta_i))\right] + \tau_i(w_i(\theta_i))$$

which is agent i's expected utility if she reports truthfully. We have shown that it is not profitable for an agent to deviate from a truthful report. □

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