Cheap Talk in Allocation Problems*

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Abstract

We consider a partially informed principal who needs to allocate a single good among multiple agents. Each agent wants to receive the good and holds partial information about the principal's payoff from allocating the object to him. Both the agents' and the principal's signals are ordered in the first-order stochastic dominance sense. We show that effective information may be transmitted from the agents to the principal via cheap-talk communication despite the fact that each agent always prefers to receive the good. We characterize the information structures that support effective information transmission, and analyze how information transmission considerations affect the optimal signal function of the principal. (Keywords: Cheap talk; Allocation problems; State-independent preferences; Information transmission.)

1 Introduction

The allocation of a single item is one of the most prominent problems in economics and corresponds to many economic applications. Examples include cases of hiring a job candidate, investing in a start-up company, allocating a grant to a research team, and so on. The problem deals with a principal who needs to allocate a single good among

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multiple agents and wants to allocate it to the agent with the highest value. Each agent wants to receive the good and has private information about his value. In order to optimize her decision, the principal wants to elicit the agents' private information. The question is whether the economic environment provides a channel for information transmission from the agents to the principal.

One channel of information transmission is signaling devices. The signaling literature, initiated in the seminal paper of Spence (1973), considers environments where information transmission is possible due to the variation of an agent's cost from using the signaling device as a function of his private information.[1] Another important environment that considers allocation problems is auctions. An auction exploits the property that an agent with a higher valuation has a higher willingness to pay in order to screen the agent's type.[2] Another way in which agents can transmit information is by providing evidence. These kinds of environments are dealt with in the literature on evidence games and verifiable disclosure.[3]

The most natural channel of information transmission, however, is direct non-costly and non-verifiable communication between the agents and the principal, namely, cheap-talk communication. There are many economic applications of allocation problems in environments where neither signaling devices nor transfers are available and where the agents' private information is not verifiable.[4] In such environments the only channel for information transmission is cheap-talk communication. Nonetheless, the literature on cheap-talk communication, initiated in the seminal paper of Crawford and Sobel (1982), has abstracted away from dealing with allocation problems and has focused on environments where the sender's and the receiver's preferences are somewhat aligned.[5] In this paper we examine an allocation problem in a setting where the agents and the principal have private partial information about the agents' values.[6] The setting

[1]See also Stiglitz (1985), for surveys see Riley (2001), Stiglitz (2002), and Spence (2002).

[2]See, e.g., Vickrey (1961), Myreson (1981), Milgrom and Weber (1982). For a comprehensive survey see Krishna (2009).

[3]See, e.g., Grossman (1981), Jovanovich (1982), Dye (1985). For a survey see Verrecchia (2001).

[4]One example is job interviews where it is reasonable to assume that the interviewee may possess relevant information that is unverifiable. In addition, it is unorthodox for money to change hands in a job interview.

[5]See, e.g., Farrell and Rabin (1996). For a survey see Sobel (2013).

[6]Other papers consider environments with a partially informed receiver but where the sender's and

we consider is natural in the sense that both the agents and the principal signals are fully ordered in terms of first order stochastic dominance, namely, signals can be either "good" or "bad." Our main result is that even in this setting, where each agent wants to receive the good and where an agent's signals are ordered ex ante as "bad" or "good," it is possible that effective information is transmitted from the agents to the principal via cheap-talk communication.

We begin by presenting the model. Then we present an example that illustrates the economic mechanism that drives the possibility of effective cheap-talk. We then characterize the conditions under which effective information transmission is possible. We find that effective information transmission is possible only if the information agents possess is less important to the principal than the information they do not hold. We then take an agent's signal function as exogenous and consider the implications of information transmission in two situations. In the first situation the principal is also facing an exogenous signal. We analyze the principal's payoff as a function of the number of agents who participate in the allocation process. We find that the principal's utility may not be monotonic in the number of agents. That is, in some cases the principal is better off choosing from a smaller set of agents. In the second situation the principal has some control over the information she receives; namely, she can choose a signal function from a fixed set of signals. We assume that the principal cannot commit to an allocation, but we allow the principal to commit to the signal function. When choosing a signal function, the principal takes into account two effects. The first is the direct information she receives from the signal about the agents' quality. The second is the information she receives from the agents in the case where the signal function induces information transmission. We find that if the number of agents is large enough, the principal will choose a signal function that induces information transmission. This signal function is different from the optimal signal function of the principal in an environment where communication between the agents and the principal is impossible. That is, the principal strictly benefits from the ability to communicate directly with the agents.

the receiver's preferences are somewhat aligned; see, e.g., de Barreda (2010), and Ishida and Takashi (2016).

Related Literature

In the first part of the paper we deal with the possibility of effective cheap-talk communication in an allocation-problem setting. There are a few other papers that deal with the possibility of cheap-talk communication in models where all types of a sender share the same ordinal preferences. One strand of these papers deals with environments where, unlike in the allocation problem we consider, the sender's signals are not ordered, namely, there are no "bad" or "good" signals ex ante. Farrell and Gibbons (1989), Kartik, Ottaviani, and Squintani (2007), and Goltsman and Pavlov (2011) deal with a sender whose preferences are based on multiple considerations, and the disclosure of information may have conflicting effects on these different considerations. Sorting occurs because different types of the sender differ in the intensity of their preferences with respect to each of these considerations. Seidmann (1990) and Watson (1996) deal with the case where the sender faces uncertainty regarding the receiver's preferences. Each type of the receiver responds differently to the sender's information in such a way that the sender's signals are not ordered from an ex ante perspective. Another strand of these papers considers the possibility of cheap-talk communication in a model with a state-independent sender; see, e.g., Chakraborty and Harbaugh (2010) and Lipnowski and Ravid (2017). In these papers the effective cheap talk equilibrium depends on the sender being indifferent between the messages that are sent in equilibrium. In an allocation problem, however, no effective information can be transmitted in an equilibrium where a state-independent sender is indifferent between the different equilibrium messages. This is because the sender's payoff is monotonic in the receiver's expected beliefs regarding his quality. Therefore, the indifference of the sender implies that every message that is sent corresponds to the same expected belief of the receiver. This belief must be the prior belief of the receiver.

In the second part of the paper we analyze the optimal learning policy for the principal. Ben-Porath, Dekel, and Lipman (2014) consider a similar problem in an environment where the principal can verify information that is held by the agents at some cost. They assume that the principal can commit both to a verification policy and to an allocation rule and that the principal can only verify information that is held by the agents, but cannot learn new information. When the principal can only verify information, the power to commit to an allocation rule is essential to the possibility of information transmission. Otherwise, each agent will transmit, independently of his true information, the message that maximizes the principal's belief. In our setting a signal function can provide information about an agent's quality that is not held by the agent. We show that in such a case effective information transmission may arise even when the principal lacks the ability to commit to an allocation rule.

When choosing a learning policy, the principal also takes into account the effect of the signal function on cheap-talk communication. Essentially, by strategically choosing a proper signal function, the principal (receiver) is persuading the agents (senders) to report truthful information. This contributes to the literature on Bayesian persuasion that focuses on persuasion from the side of the sender; see, Kamenica and Gentzkow (2011).

The rest of the paper is organized as follows. In Section 2 we present the model. In Section 3 we present an example that illustrates how effective information transmission may arise in our setting. In Section 4 we characterize the conditions under which effective information transmission is possible. In Section 5 we analyze the implications of the possibility of effective communication in two different settings. Section 6 concludes. All proofs are relegated to the Appendix.

2 The Model

There is a set $N = \{1, .., n\}$ of n agents, $n \geq 3$, who are ex-ante symmetric. There is a single indivisible good to allocate among them. We denote by ω_i the value the principal receives from allocating the good to agent i. This value depends on the quality of the agent. We assume that the agent holds partial information about his quality and we denote this information by x_i. The information about an agent's quality that is not held by the agent is denoted by y_i. For simplicity we assume that $x_i, y_i \in \{0, 1\}$ and that the agent's value depends on these arguments in a linear manner, i.e., $\omega_i(x_i, y_i) = x_i + a \cdot y_i$, where $a \geq 0$. We assume that x_i and y_i are independently distributed according to the uniform distribution and that agents' types are independent.[7] The agents communicate with the principal via costless unverifiable messages. The principal also learns information about an agent's type directly by observing a binary test (success or failure) that is monotonic in the agent's value; i.e., an agent with a higher value is more likely to succeed on the test.[8] The test is of the following form:

1	p	1
0	0	q
y_i/x_i	0	1

where $p<1$ and $q<1$ indicate the probability of success in the relevant entries. For example, the probability of type $(0,1)$ passing the test is p. The principal cannot commit to an allocation rule. That is, the principal always allocates the good to the agent who provides him with the highest expected value given his information. We assume that every agent strictly prefers receiving the object to not receiving it. In particular, we concentrate on the case where an agent's payoff is his probability of receiving the item; namely, the agents' preferences are state independent.

3 Example of Information Transmission

In this section we illustrate by an example how effective information transmission may arise in our setting. Consider $N = \{1, 2, 3\}$ and $\omega_i(x_i, y_i) = x_i + 3 \cdot y_i$ and assume that the test is of the following form: For some $p \in [0, 1)$

1	p	1
0	0	0
y_i/x_i	0	1

[7]This formulation does not preclude the existence of a correlation between the fundamental information of the agent and the rest of the information since x_i accounts for this correlation.

[8]The assumption that the tests are binary is a way to restrict attention to the situation where a test cannot reveal all the information about the agents.

The principal allocates the object to the agent with the highest expected value, and in case some agents share the highest expected value, she allocates the object to each of these agents with equal probability. We characterize the values of p for which there exists a truthful equilibrium. Given that the principal believes that the agents report the truth, the expected values she assigns to an agent as a function of his message and his test result are

$$E\left[\omega_i|s,1\right]=4,\quad E\left[\omega_i|s,0\right]=3,\quad E\left[\omega_i|f,0\right]=\frac{1-p}{2-p}\cdot 3,\quad E\left[\omega_i|f,1\right]=1$$

For example, $E\left[\omega_i|s,1\right]$ is the expected value of agent i given that he succeeded on the test and reported $x_i=1$. When an agent of type $x_i=1$ fails, the principal learns that $y_i=0$ with certainty, whereas when an agent of type $x_i=0$ fails, the principal still assigns a positive probability to the event $y_i=1$. This probability increases as p decreases. For values of p that are above $1/2$ we get that $E\left[\omega|f,1\right]>E\left[\omega|f,0\right]$. This means that regardless of the test result it is better for the agents to report $x_i=1$. Hence, a truthful equilibrium is impossible. Nonetheless, for values of p that are below $1/2$ it holds that $E\left[\omega|f,0\right]>E\left[\omega|f,1\right]$. That is, the agent would have preferred to report $x_i=0$ had he known that he would fail the test. The explanation for this result is the following. As p decreases the agent's ability to positively influence the principal's belief on y_i through his report on x_i, when the principal observes the signal f, increases. Recall that the information the agent does not know, y_i, influences his quality to a larger extent than the information he does know, x_i. It follows that for small enough p's the improvement in the principal's belief regarding y_i from reporting $x_i=0$ instead of $x_i=1$ compensates for the decrease in the principal belief regarding x_i. We get that the signal $x_i=0$, that reflects bad information regarding the agent's quality when the receiver has no information, turns to reflect good information when the principal observes the signal f.

When $p<1/2$ we get that the principal's beliefs for a given report and a test result are ordered as follows:

$$E\left[\omega_i|s,1\right]>\left[\omega_i|s,0\right]>\ E\left[\omega_i|f,0\right]>\ E\left[\omega_i|f,1\right]$$

Given this order we can calculate the probability that an agent will receive the object as a function of his report and a test result.[9] For example, $S1$ is the probability that an agent who succeeds on the test and reports $x_i = 1$ will receive the object:

$$S1 = \frac{37}{48}$$

$$S0\left(p\right) = \frac{p^2 - 9p + 27}{48}$$
$$F0\left(p\right) = \frac{p^2 - 7p + 13}{48}$$
$$F1 = \frac{1}{48}$$

Now, agents with different values of x_i have different probabilities of succeeding on the test, and so they face different lotteries. In a truthful equilibrium each agent prefers the lottery that corresponds to his true value of x_i.
The incentive compatibility constraint of an agent with $x_i = 0$ is

$$\frac{p}{2} \cdot S0\left(p\right) + \frac{2-p}{2} \cdot F0\left(p\right) \geq \frac{p}{2} \cdot S1 + \frac{2-p}{2} \cdot F1$$

Rearranging the inequality we get:

$$\frac{p}{2} \cdot \left(S0\left(p\right) - S1\right) + \frac{2-p}{2} \cdot \left(F0\left(p\right) - F1\right) \geq 0$$

Note that by construction we have that for every $p \in [0,1)$, $S0\left(p\right) - S1 < 0$ and $F0\left(p\right) - F1 > 0$. That is, given this order of the principal's beliefs the agent would have reported $x_i = 0$ had he known that he would fail the test, and he would have reported $x_i = 1$ had he known that he would be successful. Therefore, as p , the probability of type $x_i = 0$ to succeed in the test, increases, an agent of type $x_i = 0$ is ex-ante less exposed to the test result that incentivizes truthful reporting and more exposed to the test result that incentivizes misreporting . Hence, the incentive of type $x_i = 0$ to report truthfully decreases as p increases. In our particular example the IC

[9] Recall that, the principal allocates the object to the agent with the highest expected value, and in case some agents share the highest expected value, she allocates the object to each of these agents with equal probability.

constraint of agent of type $x_i = 0$ holds iff $p \in [0, \frac{2}{3}]$.
The incentive compatibility constraint of an agent with $x_i = 1$ is

$$\frac{1}{2} \cdot S0\,(p) + \frac{1}{2} \cdot F0\,(p) \le \frac{1}{2} \cdot S1 + \frac{1}{2} \cdot F1$$

The payoff of type $x_i = 1$ when he reports truthfully does not depend on p. Therefore, in order to analyze the IC constraint of type $x_i = 1$, we concentrate only on the effect of a change in p on his payoff from misreporting, i.e. reporting $x_i = 0$. As p increases agents of type $x_i = 0$ become fiercer competitors of an agent of type $x_i = 1$ who misreport. It follows that the payoff from misreporting for an agent of type $x_i = 1$ is decreasing in p. From that we can conclude that the incentive of agent of type $x_i = 1$ to report truthfully is increasing in p. In our particular example the IC constraint of agent of type $x_i = 1$ holds iff $p \in \left[4 - \sqrt{15}, 1\right)$.

We get that given the order of the principal's beliefs both IC constraints hold iff $p \in [0, \frac{2}{3}] \bigcap \left[4 - \sqrt{15}, 1\right) = [4-\sqrt{15}, \frac{2}{3}]$. Since the required order of the principal's beliefs holds iff $p \in [0, \frac{1}{2})$ we get that truthful equilibrium exists iff $p \in [4 - \sqrt{15}, \frac{2}{3}] \bigcap [0, \frac{1}{2}) = [4 - \sqrt{15}, \frac{1}{2})$.

4 Tests that Enable Information Transmission

In this section we characterize the tests that enable an effective cheap-talk equilibrium. Throughout the analysis we focus on equilibria in pure strategies, i.e., truthful equilibira. For clarity of exposition we concentrate on a subset of tests in which $q = 0$; in Appendix B we provide such a characterization when we allow q to vary.

In a truthful equilibrium a message from the agent and a realization of a test result translate to a belief of the principal regarding the agent's quality via Bayes law. The principal's beliefs are expressed as follows:

$$E\left[\omega_i | s, 1\right] = a + 1, \quad E\left[\omega_i | s, 0\right] = a, \quad E\left[\omega_i | f, 0\right] = \frac{1-p}{2-p} \cdot a, \quad E\left[\omega_i | f, 1\right] = 1$$

The principal allocates the object to the agent with the highest expected value, and in the case where some agents share the highest expected value, she allocates the object to each of these agents with equal probability. A truthful equilibrium can arise only if $E[\omega_i|f,0] \geq E[\omega_i|f,1]$. Otherwise, each agent would prefer to report $x_i = 1$ regardless of the test result. We define $\hat{p}(a) = \frac{a-2}{a-1}$ to be the maximal p such that given a truthful report we have $E[\omega_i|f,0] \geq E[\omega|f,1]$ and we state the following claim:

Claim 1. A truthful equilibrium exists only if $p < \hat{p}(a)$.

A smaller p corresponds to a stronger correlation between the test result and the agent's information. The interpretation of Claim 1 is that effective cheap-talk can arise only if this correlation is sufficiently strong.
Since $0 \leq p$ we can also deduce the following corollary.

Corollary 2. *A truthful equilibrium exists only if* $a \geq 2$.

The interpretation of Corollary 2 is that for a truthful equilibrium to occur, the agent need not hold too much information about his quality.

Remark. For cheap-talk to arise in environments with a state independent sender and a partially informed receiver, no message of the sender should map a better action for the sender regardless of the receiver's information.[10] This property emerges exogenously when the receiver has private information about his preferences; see, e.g., Watson (1996). In our model the receiver's preferences are commonly known and this property emerges endogenously.

We now turn to find properties of tests that allow for a truthful equilibrium given that the principal allocates the object according to the following order, $\succ_*$:

$$\text{(success,1)} \succ_* \text{(success,0)} \succ_* \text{(failure,0)} \succ_* \text{(failure,1)}$$

i.e, the principal prefers to allocate the item to an agents who succeeded in the test and reported $x_i = 1$ than to an agent who succeeded in the test and reported $x_i = 0$ and so on. We calculate the probability that an agent will receive the object as a function of his report, the number of agents, and the test result. For example, $S1(n,p)$ is the probability that an agent who succeeds on the test and reports $x_i = 1$ will receive the object, given the test p and the number of agents n:

[10] This property is also necessary when all types of the agent share the same ordinal preferences.

$$S1\left(n,p\right)=1-\left(\sum_{i=1}^{n-1}\left(\begin{array}{c}n-1\\ i\end{array}\right)\cdot\frac{i}{i+1}\cdot\left(\frac{1}{4}\right)^{i}\cdot\left(\frac{3}{4}\right)^{n-i-1}\right)$$

$$S0\left(n,p\right)=1-\left(\sum_{i=1}^{n-1}\left(\begin{array}{c}n-1\\ i\end{array}\right)\left(\frac{1}{4}\right)^{i}\cdot\left(\frac{3}{4}\right)^{n-i-1}\right)-\left(\sum_{i=1}^{n-1}\left(\begin{array}{c}n-1\\ i\end{array}\right)\cdot\frac{i}{i+1}\cdot\left(\frac{p}{4}\right)^{i}\cdot\left(\frac{3-p}{4}\right)^{n-i-1}\right)$$

$$F0\left(n,p\right)=\sum_{i=1}^{n-1}\left(\begin{array}{c}n-1\\ i\end{array}\right)\cdot\frac{1}{i+1}\cdot\left(\frac{2-p}{4}\right)^{i}\cdot\left(\frac{1}{4}\right)^{n-i-1}+\left(\frac{1}{4}\right)^{n-1}$$

$$F1\left(n,p\right)=\frac{1}{n}\cdot\left(\frac{1}{4}\right)^{n-1}$$

Each agent will find it optimal to report truthfully given the principal's beliefs if and only if the following constraints hold:

$$\text{IC of } x_i\text{=0:}\qquad \frac{p}{2}\cdot S0\left(n,p\right)+\frac{2-p}{2}\cdot F0\left(n,p\right)\geq\frac{p}{2}\cdot S1\left(n\right)+\frac{2-p}{2}\cdot F1\left(n\right)$$

$$\text{IC of } x_i\text{=1:}\qquad \frac{1}{2}\cdot S0\left(n,p\right)+\frac{1}{2}\cdot F0\left(n,p\right)\leq\frac{1}{2}\cdot S1\left(n\right)+\frac{1}{2}\cdot F1\left(n\right)$$

For every $x_i\in\{0,1\}$ we define $IC_{x_i}\left(n\right)$ to be the set of p's for which the IC constraint of type x_i holds and $p_{x_i}\left(n\right)$ to be the p for which the IC constraint of type x_i holds in equality.

Lemma 3. *$IC_0\left(n\right)=\left[0,p_0\left(n\right)\right]$ and $IC_1\left(n\right)=\left[p_1\left(n\right),1\right)$*

The result on $IC_0\left(n\right)$ follows from the fact that the expected payoff of $x_i=0$ given a truthful report does not depend on p, whereas the incentive of $x_i=0$ to misreport is increasing in p because $p_r(s|0)=\frac{p}{2}$. The result on $IC_1\left(n\right)$ follows from the fact that the expected payoff of $x_i=1$ given a truthful report does not depend on p, whereas the incentive of $x_i=1$ to misreport is decreasing in p because both $S0\left(n,p\right)$ and $F0\left(n,p\right)$ are decreasing in p.

Lemma 4. *$p_1\left(n\right)<p_0\left(n\right)$ for every $n\geq3$.*

Given the principal's order $\succ_*$, an agent wants to report $x_i=1$ in the case of a success and $x_i=0$ in the case of a failure irrespective of his type. The result follows from the fact that the probability of success for type $x_i = 1$ is higher than that of type $x_i = 0$. This means that for each $p \in [0, 1)$ for which type $x_i = 1$ weakly prefers to misreport, type $x_i = 0$ strongly prefers to report truthfully; and for each $p \in [0, 1)$ for which type $x_i = 0$ weakly prefers to misreport, type $x_i = 1$ strongly prefers to report truthfully.

We now characterize the set of tests in which a truthful equilibrium exists. These tests have to satisfy both the condition about the appropriate order, i.e., $p < \hat{p}(a)$, and the incentive-compatibility conditions.

Theorem 5. *Assume that $a > 2$. There exist two decreasing functions $p_1(n) < p_0(n)$, where $0 < p_0(n)$ for every $3 \leq n$, and there exist $n' < n''$ such that effective cheap-talk arises if and only if*

$$p \in A := [p_1(n), p_0(n)] \cap [0, \hat{p}(a)).$$

If $n \leq n'$ then A is empty. $(\hat{p}(a) \leq p_1(n'))$

If $n' < n < n''$ then $A = [p_1(n), \hat{p}(a))$

If $n'' < n$ then $A = [p_1(n), p_0(n)]$.

As n increases the probability that there exists an agent of type $(s, 1)$ increases. Hence, the incentive to report $x_i = 1$ increases. This implies that $IC_0(n+1) \subset IC_0(n)$ and $IC_1(n) \subset IC_1(n+1)$. The result on the monotonicity of $p_1(n)$ and $p_0(n)$ follows from Lemma 3. At $p = 0$ the IC of $x_i = 0$ holds strictly; from continuity we get that $0 < p_0(n)$ for every $3 \leq n$.

5 The Implications of Information Transmission

In this section we show how the existence of a truthful equilibrium can affect the principal's preferences regarding the economic environment she is facing. We consider two situations. In the first situation there is no cheap-talk phase prior to the test, while in the second situation there is. We compare between these situations in two economic environments. In the first environment the principal is facing a fixed test, while in the second environment the principal can choose a test. We find that if the principal is facing a fixed test then when there is no cheap-talk phase the principal will always prefer that more agents participate in the allocation process. However, when there is a cheap-talk phase the principal's payoff may not be monotonic in the number of agents. That is, for some tests the principal will prefer that fewer agents participate in the allocation process despite the fact that the tests are costless. When the principal can choose the test we find that if the number of agents exceeds some threshold, then the test the principal chooses when there is a cheap-talk phase is different from the test she chooses when there is no cheap-talk phase. That is, the principal gains a strictly higher payoff if she communicates with the agents.

5.1 A Principal Facing a Fixed Test

Assume the principal can perform a single test

1	p	1
0	0	0
y/x	0	1

where $p \in [0,1]$. Consider the situation where there is no cheap-talk communication. In this case the principal's payoff is strictly increasing in the number of agents n. That is, the larger the number of agents who participate in the allocation process is, the higher the expected value of the agent who will receive the object is. To see this note that the expected value of the highest ranked agent is

$$\left(1-\left(\frac{3-p}{4}\right)^{n}\right)\cdot E\left(\omega_i|S\right)+\left(\frac{3-p}{4}\right)^{n}\cdot E\left(\omega_i|F\right)$$

since $E\left(\omega_i|S\right) > E\left(\omega_i|F\right)$ this expression is increasing in n.

We now consider the case where the agents communicate with the principal prior to the test. We first show that given a fixed test p the existence of a truthful equilibrium depends on the number of agents.

Claim 6. For every $p < \hat{p}(a)$ there exist $\underline{n}(p) \leq \overline{n}(p)$ such that a truthful equilibrium exists if and only if $\underline{n}(p) \leq n \leq \overline{n}(p)$.

In Theorem 5 we saw that the set of tests for which a truthful equilibrium exists $[p_1(n), p_0(n)]$ depends on the number of agent. Fix a test p. We get that for $n < \underline{n}(p)$ we have $p < p_1(n)$; for $\underline{n}(p) \leq n \leq \overline{n}(p)$ we have $p_1(n) \leq p \leq p_0(n)$; and for $\overline{n}(p) < n$ we have $p_0(n) < p$.

We now show that the principal's payoff may not be monotonic in the number of agents. To see this consider the following example. Assume that $a = 2.5$ and the principal is facing the test $p = \frac{11}{51}$; in this case we have that $\overline{n}\left(\frac{11}{51}\right) = 6$. That is, if the number of agents is larger than 6 then there is no truthful equilibrium. The principal's payoff when $n = 6$ is larger than her payoff when $n = 7$. The intuition for this result is as follows. The increase of n from 6 to 7 implies that a truthful equilibrium ceases to exist. Without a truthful equilibrium the principal can distinguish between agents based only on the results of the test: success or failure. On the other hand, a truthful equilibrium allows the principal to distinguish between agents within these categories. Increasing the number of agents improves the expected ability of the best agent. However, the identification of the best agent is more accurate under a truthful equilibrium.

5.2 A Principal who Chooses the Test

Assume that the principal can choose any test. That is, she can choose any $p \in [0,1)$. Consider the situation where there is no cheap-talk communication. We now characterize the properties of the optimal test as a function of the number of agents n.

Claim 7. For every $a > 2$, the optimal test $p(n,a)$ is a decreasing function of n and there exists $n_0(a)$ such that for every $n_0(a) \leq n$ we have $p(n,a) = 0$.

The explanation for this result is as follows. As the number of agents increases the probability of the event that at least one of the agents is of the best type (1,1) increases. Therefore, the principal will want to choose the test that accurately identifies type (1,1).

We now consider the case where the agents communicate with the principal prior to the test. We show that for every n we have that the optimal test $p(n,a) > 0$. That is, the optimal test where there is a cheap-talk communication is different than the optimal test in the case where there is no cheap-talk communication.

Theorem 8. *Consider $a > 2$. For every* $\max\{n_0(a), n'\} \leq n$ *the principal will choose the maximal p that allows for truthful equilibrium, i.e.,* $p(n, a) = \min\{\hat{p}(a), p_0(n)\}$, *which is greater than 0 for every $n \geq 3$.*

For $n_0(a) \leq n$, the optimal test without cheap-talk, $p = 0$, identifies the best type (and pools all other types). Given a test $p > 0$ that allows for a truthful equilibrium the best type (1,1) is identified and since p is positive there is also a partial identification of the second-best type (0,1). That is, choosing the maximal p that allows for a truthful equilibrium dominates the test $p = 0$. This result implies that by communicating with the agents the principal can achieve a strictly higher payoff. The following table shows values of $n_0(a)$ and $p_0(n_0(a))$ for several values of a:

a	$\hat{p}(a)$	$n_0(a)$	$p_0(n_0(a))$
2.5	0.33	6	0.21
3	0.5	5	0.31
4	0.66	8	0.11

6 Conclusion

We have considered the possibility of effective cheap-talk communication in an allocation problem environment where both the agents and the principal have partial information about the agents' values and their signals are ordered in terms of first-order stochastic dominance. Our analysis yields two important insights. The first is that effective cheap-talk can arise in unidimensional environments with state-independent senders and a receiver whose preferences are commonly known. The second is that learning procedures may also have a secondary strategic effect in inducing agents to report their information truthfully. This effect should be incorporated into the consideration system of the principal when she chooses a learning policy. The environment we've considered corresponds to many real-life applications such as hiring procedures in workplace environments, investment in a start-up company by a venture-capital firm, and many more.

Appendix A

Proofs

Proof of Claim 1.

Assume by contradiction that $p < \hat{p}(a)$ and truthful equilibrium exists. If $p < \hat{p}(a)$ then $E[\omega_i \mid f, 0] < E[\omega_i \mid f, 1]$, so in case of a failed test it is best to report $m = 1$. It is always the case that in case of a successful test the preferred report is $m = 1$, so we get that no matter what is the result of the test the preferred report is $m = 1$. It follows that ex-ante the preferred report is $m = 1$, this is a contradiction to the existence of a truthful equilibrium.$\blacksquare$

Proof of Corollary 2.

First note that $\forall p \in [0, 1]$ $\frac{1-p}{2-p} = 1 - \frac{1}{2-p} \leq \frac{1}{2}$, it follows that if $a < 2$ then $\frac{1-p}{2-p}a = E[\omega_i \mid f, 0] < E[\omega_i \mid f, 1] = 1$. We get that $a \geq 2$ is a necessary condition for $E[\omega_i \mid f, 0] \geq E[\omega_i \mid f, 1]$ and we have already established that $E[\omega_i \mid f, 0] \geq E[\omega_i \mid f, 1]$ is a necessary condition for the existence of a truthful equilibrium.$\blacksquare$

Proof of Lemma 3.

First note that given a truthful equilibrium the expected payoff of an agent, if he follows the equilibrium strategy, i.e., reports the truth, does not depend on the value of p. This observation is clear in the case of $x = 1$, but it is also true if $x = 0$. The reason is the following, an agent with $x_i = 0$ will always be ranked bellow an agent with $x = 1$ that succeeded in the test and always be ranked above an agent with $x = 1$ that failed in the test, regardless of the result of the agent with $x = 0$. So the parameter p only determine the internal distribution of wins among agent with $x = 0$, but it is obvious that from an ex-ante perspective the probability that one agent with $x = 0$ would outrank another agent with $x = 0$ is $\frac{1}{2}$ no matter what the value of p is. From this we get that in order to prove the part of lemma that deals with agent with $x_i = 0$ it is enough to show that the expected payoff in case of misreporting is increasing in p. This is clear because an increase in p corresponds to an increase in the probability of the event in which misreporting is better. Similarly, in order to prove the part of lemma that deals with agent with $x_i = 1$ it is enough to show that the expected payoff in case of misreporting is decreasing in p. This is true because the probability it would outrank an agent of type $x = 1$ does not depend on p, but the probability it would outrank an agent of type $x = 0$ depends negatively on p.■

Proof of Lemma 4.

If $p = p_0(n)$ an agent of type $x = 0$ is indifferent between reporting the truth, i.e., $m = 0$, and misreporting ,i.e., $m = 1$. If $p < 1$ then the probability of a successful test is larger for type $x = 1$. Remember that in a truthful equilibrium we have that an agent wants to report $m = 1$ in case of a successful test and $m = 0$ in case of a failure irrespective of his type, and that the payoff given a report and a result of the test does not depend on the type. It follows that if type $x = 0$ is indifferent, type $x = 1$ must strictly prefer to report $m = 1$, i.e., the truth. From that we get that $p_0(n) \in intIC_1(n)$, it follows from lemma 3 that $p_0(n) > p_1(n)$.■

Proof of Theorem 5.

First note that we can rewright $S1(n), S0(n,p), F0(n,p), F1(n,p)$in the following way:

$S1(n) = E\left[\frac{1}{x+1}\right]$ when $x \sim Bin\left(n-1, \frac{1}{4}\right)$

$S0(n) = \left(\frac{3}{4}\right)^{n-1} \cdot E\left[\frac{1}{x+1}\right]$ when $x \sim Bin\left(n-1, \frac{p}{3}\right)$

$F0(n) = \left(\frac{3-p}{4}\right)^{n-1} \cdot E\left[\frac{1}{x+1}\right]$ when $x \sim Bin\left(n-1, \frac{2-p}{3-p}\right)$

$F1(n) = \left(\frac{1}{4}\right)^{n-1} \cdot \frac{1}{n}$ when $x \sim Bin\left(n-1, \frac{1}{4}\right)$

It is clear that if we fix $p \in [0,1)$ and let n grow, the decay of $S1(n)$ is polynomial while the decay of $S0(n,p), F0(n,p), F1(n,p)$ is exponential. From that we can deduce that $p_1(n)$ is strictly decreasing and that there exist $N \in \mathbb{N}$ such that for every $n > N$ the IC constraint of type $x_i = 1$ holds for every $p \in [0,1)$,that is for every $n > N$ it holds that $p_1(n) = 0$. If we look at the IC constraint of type $x_i = 0$ it is clear from the same reason that $p_0(n)$ must behave approximately like the function $\left(\frac{3}{4}\right)^n$, this function is obviously decreasing.

At $p = 0$ the IC of type $x = 0$ holds strictly, from continuity we get that $p_0(n) > 0$ for every $n \geq 2$. The characterization result is a direct consequence of the fact that $p_0(n)$ is strictly decreasing in n, and $p_1(n)$ is strictly decreasing until it reaches zero.■

Proof of Claim 6.

A direct consequence of the characterization result in Theorem 5 is that if we fix a test, i.e., fix $p < \hat{p}(a)$, then we can potentially have a range $n < \underline{n}(p)$ where $p < p_0(n)$, an intermediate range $\underline{n}(p) \leq n \leq \overline{n}(p)$ where $p_0(n) \leq p \leq p_1(n)$ and finally a range $\overline{n}(p) < n$ where $p_1(n) < p$. As we have established earlier, effective cheap talk arises only at the intermediate range.

Proof of Claim 7.

We start the proof by introducing the exact expression for the expected quality of the chosen agent:

$$E\left[\omega_{chosen}\right](p,a,n) = \left(1 - \left(\frac{3-p}{4}\right)^n\right)\left(\frac{1}{1+p} + a\right) + \left(\frac{3-p}{4}\right)^n \left(\frac{1+a\left(1-p\right)}{3-p}\right)$$

We can rewrite this expression in the next manner

$$E\left[\omega_{chosen}\right](p,a,n) = \left(\frac{1}{1+p} + a\right) + \left(\frac{3-p}{4}\right)^n \left(\frac{1+a\left(1-p\right)}{3-p} - \left(\frac{1}{1+p} + a\right)\right)$$

.

We will prove the claim by showing that there exists $N \in \mathbb{N}$ such that for every $n > N$, $p \in [0,1]$ and $a > 0$ it holds that

$$\frac{\partial E\left[\omega_{chosen}\right]}{\partial p}(p,a,n) =$$

$$-\frac{1}{(1+p)^2} + \left(\frac{3-p}{4}\right)^n \left(\frac{-a\left(3-p\right) - \left(1+a\left(1-p\right)\right)}{\left(3-p\right)^2} + \frac{1}{(1+p)^2}\right) - n\left(\frac{3-p}{4}\right)^{n-1}\left(\frac{1+a\left(1-p\right)}{3-p} - \left(\frac{1}{1+p} + a\right)\right) < 0$$

If this is true it follows that for $n > N$ the optimal p is zero. The first expression, $-\frac{1}{(1+p)^2}$ does not depend on n, and it is negative for all $p \in [0,1]$.
If we will show that

$$\lim_{n\to\infty}\left(\frac{3-p}{4}\right)^n \left(\frac{-a\left(3-p\right) - \left(1+a\left(1-p\right)\right)}{\left(3-p\right)^2} + \frac{1}{(1+p)^2}\right) - n\left(\frac{3-p}{4}\right)^{n-1}\left(\frac{1+a\left(1-p\right)}{3-p} - \left(\frac{1}{1+p} + a\right)\right) = 0$$

it will end the proof.
It is clear that

$$\lim_{n\to\infty}\left(\frac{3-p}{4}\right)^n \left(\frac{-a\left(3-p\right) - \left(1+a\left(1-p\right)\right)}{\left(3-p\right)^2} + \frac{1}{(1+p)^2}\right) = 0$$

because $0 < \frac{3-p}{4} < 1$ for all $p \in [0,1]$, therefore $\lim_{n\to\infty} \left(\frac{3-p}{4}\right)^n = 0$, the equality follows from the fact that $\left(\frac{-a(3-p)-(1+a(1-p))}{(3-p)^2} + \frac{1}{(1+p)^2}\right)$ does not depend on n.
We now move to

$$\lim_{n\to\infty} n\left(\frac{3-p}{4}\right)^{n-1}\left(\frac{1+a\,(1-p)}{3-p} - \left(\frac{1}{1+p}+a\right)\right) = 0$$

because $\left(\frac{1+a(1-p)}{3-p} - \left(\frac{1}{1+p}+a\right)\right)$ does non depend on n it is sufficient to show that

$$\lim_{n\to\infty} n\left(\frac{3-p}{4}\right)^{n-1} = 0$$

Denote $a_n = \left(\frac{3-p}{4}\right)^{n-1}$. The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{3-p}{4}\right)^{n-1}$ is converging as a geometric series with a common ratio between -1 and 1.
From "Cauchy convergence test" we have that for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that for every $n > N$ and for every natural $s \geq 1$ it holds that $|a_{n+1} + a_{n+2} + \cdots + a_{n+s}| < \varepsilon$.
Additionally, we have that the sequence $\{a_n\}_{n=1}^{\infty}$is strictly decreasing.
Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that for every $n > N$ and for every natural $s \geq 1$: $s \cdot a_{n+s} < a_{n+1} + a_{n+2} + \cdots + a_{n+s} < \frac{\varepsilon}{2}$. Given $m > 2 \cdot N + 2$ we choose $n \in \mathbb{N}$ such that $N < n < \frac{m}{2} < m$ and also choose $s = m - n$, so $a_{n+s} = a_m$. We get that $\frac{m}{2} \cdot a_m < s \cdot a_{n+s} < \frac{\varepsilon}{2}$, it follows that $m \cdot a_m < \varepsilon$. This argument establishes that $\lim_{n\to\infty} n\left(\frac{3-p}{4}\right)^{n-1}\left(\frac{1+a(1-p)}{3-p} - \left(\frac{1}{1+p}+a\right)\right) = 0$, and so it follows that:

$$\lim_{n\to\infty}\left(\frac{3-p}{4}\right)^n\left(\frac{-a\,(3-p)-(1+a\,(1-p))}{(3-p)^2}+\frac{1}{(1+p)^2}\right) - n\left(\frac{3-p}{4}\right)^{n-1}\left(\frac{1+a\,(1-p)}{3-p}-\left(\frac{1}{1+p}+a\right)\right) = 0$$

From this we derive that there exists $N \in \mathbb{N}$ such that for every $n > N$, $p \in [0,1]$, and $a > 0$ it holds that:

$$\frac{\partial E\left[\omega_{chosen}\right]}{\partial p}(p,a,n) =$$

$$-\frac{1}{(1+p)^2}+\left(\frac{3-p}{4}\right)^n\left(\frac{-a\,(3-p)-(1+a\,(1-p))}{(3-p)^2}+\frac{1}{(1+p)^2}\right)-n\left(\frac{3-p}{4}\right)^{n-1}\left(\frac{1+a\,(1-p)}{3-p}-\left(\frac{1}{1+p}+a\right)\right) < 0$$

■

Proof of Theorem 8.

From Theorem 5 we have that there exists $n'(a) \in \mathbb{N}$ such that for every $n > n'(a)$ it holds that $A(n,a) \neq \phi$. From claim 7 we have that there exists $n_0(a) \in \mathbb{N}$ such that for every $n > n_0(a)$ the optimal test in the model without cheap-talk is $p(n,a) = 0$. Note that for $n > n_0(a)$ the principal only gains from selecting $p \in A(n,a)$. If the principal selects $p = 0$ she will surely identify agents with a value $(1,1)$ if there are any in the profile, but she will not be able to distinguish between agents with a different value. That is, if there is no agent with a value $(1,1)$ in the profile the principal will have to chose an agent at random. Alternatively, if the principal selects $0 \neq p \in A(n,a)$ she will also surely identify agents with value $(1,1)$. In this case an agent with a value $(1,1)$ will surely be successful in the test and he will also send the message $m = 1$. From the successful test the principal will learn that $y_i = 1$, and from the message the principal will learn that $x_i = 1$. The important point is that in the case where the principal selects $0 \neq p \in A(n,a)$ the principal would be able to learn more refined information about agents with values that are different from $(1,1)$. This will enable the principal to select an agent in a more suited way if there are no agents of type $(1,1)$ in the profile.■

Appendix B

First note that if we allow q to vary then we get that in a truthful equilibrium the beliefs of the principal are as follows:

$$E[\omega_i \mid S,1] = 1 + \frac{a}{1+q}\ ,\ E[\omega_i \mid S,0] = a\ ,\ E[\omega_i \mid f,0] = \frac{1-p}{2-p} \cdot a\ ,\ E[\omega_i \mid f,1] = 1$$

As we saw earlier a necessary condition for truthful equilibrium is:

$$E[\omega_i \mid S,1] \geq E[\omega_i \mid S,0]\ ,\ E[\omega_i \mid f,0] \geq E[\omega_i \mid f,1]$$

.

If $q = 0$ this condition is equivalent to: $p \leq \hat{p}(a) = \frac{a-2}{a-1}$.
If we allow q to vary then this condition is equivalent to: $p \leq \hat{p}(a) = \frac{a-2}{a-1}$ and $q \leq \hat{q}(a) = \frac{1}{a}$.
Given the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$ let us explicitly formulate the expected payoff (probability of winning the item) in a truthful equilibrium as a function of the test result and the report.

$$S1(n,p,q) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \frac{i}{i+1} \cdot \left(\frac{1+q}{4}\right)^i \cdot \left(\frac{3-q}{4}\right)^{n-1-i} \right)$$

$$S0(n,p,q) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \left(\frac{1+q}{4}\right)^i \cdot \left(\frac{3-q}{4}\right)^{n-1-i} \right) - \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \frac{i}{i+1} \cdot \left(\frac{p}{4}\right)^i \cdot \left(\frac{3-q-p}{4}\right)^{n-1-i} \right)$$

$$F0(n,p,q) = \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \frac{1}{i+1} \cdot \left(\frac{2-p}{4}\right)^i \cdot \left(\frac{1-q}{4}\right)^{n-1-i} \right) + \left(\frac{1-q}{4}\right)^{n-1}$$

$$F1(n,p,q) = \frac{1}{n} \cdot \left(\frac{1-q}{4}\right)^{n-1}$$

We now move to a necessary and sufficient condition for a truthful equilibrium given the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$, i.e., the incentive compatibility constraints.

IC of $x = 0$:

$$\frac{p}{2} \cdot S0(n,p,q) + \frac{2-p}{2} \cdot F0(n,p,q) \geq \frac{p}{2} \cdot S1(n,p,q) + \frac{2-p}{2} \cdot F1(n,p,q)$$

IC of $x = 1$:

$$\frac{1+q}{2} \cdot S1(n,p,q) + \frac{1-q}{2} \cdot F1(n,p,q) \geq \frac{1+q}{2} \cdot S0(n,p,q) + \frac{1-q}{2} \cdot F0(n,p,q)$$

For every $n \geq 3$ and $q \in [0,1)$ we define $p_x(n,q)$ to be the to be the p for which the IC constraint of type x holds in equality. Similarly, For every $n \geq 3$ and $p \in [0,1)$ we define $q_x(n,p)$ to be the to be the p for which the IC constraint of type x holds in equality.

Lemma 3a. Given the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$: $IC_0(n,q) = [0, p_0(n,q)]$ and $IC_1(n,q) = [p_1(n,q), 1)$.

Proof. The proof for the case $q = 0$ holds.

Lemma 3b. Given the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$: $IC_0(n,p) = [0, q_0(n,p)]$ and $IC_1(n,p) = [q_1(n,p), 1)$.

Proof. Given a truthful equilibrium and the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$, denote by $R := \{(S,1),(S,0),(F,1),(F,0)\}$ the set of possible pairs of test result and report. Denote by $r_i \in R$ the pair that corresponds to agent i, and by r_{-i} the profile of pairs of all agent but agent i. Let R_{-i} be the set of all possible profiles of pairs of agents other then i, we can partition this set to two sub-sets according to the next criterion: Let $R_{-i}^{x,m}$ be the set that contain a profile $r_{-i} \in R_{-i}$ if and only if agent of type $x \in \{0,1\}$ will prefer to report $m \in \{0,1\}$ (before he knows the result of the test) had he knew that the profile of the other agents will be r_{-i}. It is easy to see that $R_{-i}^{0,0} = R_{-i}^{1,0} = \{r_{-i} \in R_{-i} \mid \forall j \in -i \; r_j \in \{(F,1),(F,0)\}\}$. Take $q_1 > q_2$, it is clear that for every $r_{-i} \in R_{-i}^{0,0} = R_{-i}^{1,0}$ it holds that $p_r[r_{-i} \mid q_1] \leq p_r[r_{-i} \mid q_2]$ (with strict inequality for every profile but the profile $((F,0),(F,0),\ldots,(F,0))$), it follows that if an agent would want to report $m=0$ when $q=q_1$ he would also want to report $m=0$ when $q=q_2$, and if an agent would not want to report $m=0$ when $q=q_2$ he also would not want to report $m=0$ when $q=q_1$. The lemma follows. ■

Lemma 4a. Given the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$: $\forall q \in (0,1)$ $\forall a \geq 2$ $\forall n \geq 3$ $p_1(n,q) < p_0(n,q)$.

Proof. The proof for the case $q=0$ holds.

Lemma 4b. Given the order $E[\omega_i \mid S,1] > E[\omega_i \mid S,0] > E[\omega_i \mid f,0] > E[\omega_i \mid f,1]$: $\forall p \in (0,1)$ $\forall a \geq 2$ $\forall n \geq 3$ $q_1(n,p) < q_0(n,p)$.

Proof. The same logic of the proof for the case $q=0$ holds.

Theorem 5a. $p_1(n,q), p_0(n,q), q_1(n,p), q_0(n,p)$ are decreasing in both their arguments and $\forall q \in (0,1)$ $p_0(n,q) > 0$.

Proof. Denote $A_p(n,q) := [p_1(n,q), p_0(n,q)] \cap [0, \hat{p}(a)]$, $A_q(n,p) := [q_1(n,p), q_0(n,p)] \cap [0, \hat{q}(a)]$.

Given $a > 2$ and $n \geq 3$, a test (p,q) induces truthful equilibrium if and only if $p \in A_p(n,q)$ and $q < \hat{q}(a)$ or $q \in A_q(n,p)$ and $p < \hat{p}(a)$.

Proof. From the same logic of the $q=0$ case we get that $\forall q \in (0,1)$ $p_1(n,q), p_0(n,q)$ are decreasing in n . From the same logic of lemma 3b we get that $\forall n \geq 3$ $p_1(n,q), p_0(n,q)$ are decreasing in q .

The probability of the event $r_{-i} \in R_{-i}^{0,0} = R_{-i}^{1,0}$ is clearly decreasing in n, from that we get that $\forall p \in (0,1)$ $q_1(n,p), q_0(n,p)$ are decreasing in n. From the same logic the probability of the event $r_{-i} \in R_{-i}^{0,0} = R_{-i}^{1,0}$ is clearly decreasing in p, from that we get that $q_1(n,p), q_0(n,p)$ are decreasing in p.

A test (p,q) induces truthful equilibrium if the IC constraints holds ($p \in [p_1(n,q), p_0(n,q)]$ or $q \in [q_1(n,p), q_0(n,p)]$) and the order constraint hold ($q < \hat{q}(a)$ and $p < \hat{p}(a)$).

Claim 7a. For every $a > 1$ there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ the optimal test in the game without communication $\left(p^{NC}(n,a), q^{NC}(n,a)\right)$ is the test $(0,0)$.
Proof. Denote the expected quality of the chosen agent by $E[w_{chosen}](p,q,a,n)$.

$$E[w_{chosen}](p,q,a,n) = \left(1 - \left(\frac{p+q+1}{4}\right)^n\right) \cdot E[w_{chosen} \mid S](p,q,a) + \left(\frac{p+q+1}{4}\right)^n \cdot E[w_{chosen} \mid F](p,q,a)$$

If we take the derivative of this expression with regrades to q we will get:

$$\frac{\partial E[w_{chosen}](p,q,a,n)}{\partial q} =$$

$$\frac{\partial E[w_{chosen} \mid S](p,q,a)}{\partial q} - \frac{\partial E[w_{chosen} \mid S](p,q,a)}{\partial q} \cdot \left(\frac{p+q+1}{4}\right)^n - \frac{n}{4} \cdot \left(\frac{p+q+1}{4}\right)^{n-1} \cdot E[w_{chosen} \mid S](p,q,a) +$$

$$\frac{\partial E[w_{chosen} \mid F](p,q,a)}{\partial q} \cdot \left(\frac{p+q+1}{4}\right)^n + \frac{n}{4} \cdot \left(\frac{p+q+1}{4}\right)^{n-1} \cdot E[w_{chosen} \mid F](p,q,a)$$

From the same logic of the proof of claim 7 in the $q = 0$ case:

$$\lim_{n\to\infty} -\frac{\partial E[w_{chosen} \mid S](p,q,a)}{\partial q} \cdot \left(\frac{p+q+1}{4}\right)^n - \frac{n}{4} \cdot \left(\frac{p+q+1}{4}\right)^{n-1} \cdot E[w_{chosen} \mid S](p,q,a) +$$

$$\frac{\partial E[w_{chosen} \mid F](p,q,a)}{\partial q} \cdot \left(\frac{p+q+1}{4}\right)^n + \frac{n}{4} \cdot \left(\frac{p+q+1}{4}\right)^{n-1} \cdot E[w_{chosen} \mid F](p,q,a) = 0$$

If $a > 1$ we get that for every $p \in [0,1]$ it must be the case that $\frac{\partial E[w_{chosen}|S](p,q,a)}{\partial q} < 0$. This is because $E[w_{chosen} \mid S](p,q,a) = \frac{\frac{p}{4}\cdot a + \frac{q}{4}\cdot 1 + \frac{1}{4}\cdot(1+a)}{\frac{p+q+z}{4}}$ is a weighted average of three values $1 < a < 1 + a$, and a bigger q means a bigger weight on the smallest value (1).
It follows that for every $a > 1$ and $p \in [0,1]$ there exists $n(p) \in \mathbb{N}$ such that for every $n > n(p)$ it holds that $q^{NC}(n,a) = 0$. Now the claim follows from claim 7 for the $q = 0$ case and $n_0 = max\{n(p=0), n(q=0)\}$.■

Theorem 8a. Let $a > 2$. There exists $N \in \mathbb{N}$ such that for every $n > N$ the optimal test is $(p^*(n,a), 0)$ where $p^*(n,a)$ is the maximal $p \in [0,1]$ that enables truthful equilibrium given that $q = 0$, i.e., $p^*(n,a) = min\{\hat{p}(a), p_0(n,0)\}$ which is greater than 0 for every n.
Proof. Denote the expected quality of the chosen agent given a truthful equilibrium by $E^{CT}[w_{chosen}](p,q,a,n)$.

$$E^{CT}[w_{chosen}](p,q,a,n) =$$

$$\left(1 - \left(\frac{3-q}{4}\right)^n\right) \cdot E[w_{chosen} \mid S,1](p,q,a) + \left(\left(\frac{3-q}{4}\right)^n - \left(\frac{3-q-p}{4}\right)^n\right) \cdot E[w_{chosen} \mid S,0](p,q,a) +$$

$$\left(\left(\frac{3-q-p}{4}\right)^n-\left(\frac{1-q}{4}\right)^n\right)\cdot E\left[w_{chosen}\mid F,0\right](p,q,a)+\left(\left(\frac{1-q}{4}\right)^n\right)\cdot E\left[w_{chosen}\mid F,1\right]$$

From the the same logic of the proof of claim 7 in the $q=0$ case we get that :

$$\lim_{n\to\infty}\frac{\partial E^{CT}\left[w_{chosen}\right](p,q,a,n)}{\partial q}=\frac{\partial E\left[w_{chosen}\mid S,1\right](p,q,a)}{\partial q}=\frac{\partial\left(1+\frac{a}{1+q}\right)}{\partial q}=-\frac{a}{(1+q)^2}$$

It holds that for every $q\in[0,1]$ $-\frac{a}{(1+q)^2}<0$, it means that given a truthful equilibrium and n large enough it must hold that $q=0$ in the optimal test.

From that we get that if the principal must choose a test that enables truthful equilibrium and n is large enough the principal will choose the maximal p that allows for truthful equilibrium given that $q=0$.

Note that from claim 7 we have that if we are in a scenario without communication and n is large enough the principal will choose the test $\left(p^{NC}(n,a)=0,q^{NC}(n,a)=0\right)$.

It is easy to see that the expected quality of the chosen agent is bigger in the scenario where the principal must choose a test that enables truthful equilibrium.

It follows that for n large enough the principal will choose to enable truthful equilibrium, the theorem follows.■

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