Optimal Allocation with Partial Commitment*

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Abstract

We consider a principal that needs to allocate a single good among multiple agents. Each agent wants to receive the good and holds partial information about the principal's payoff from allocating the object to him. There are no monetary transfers. The principal can commit to a test that reveals partial information about her payoff from allocations. However, the principal cannot commit to an allocation rule. We show that although the principal lacks the ability to commit to an allocation rule and despite the fact that agents hold state-independent preferences, effective information may be transmitted from the agents to the principal via cheap talk communication. We characterize the information structures that support effective information transmission, and analyze how information transmission considerations affect the optimal test choice of the principal.

1 Introduction

Consider a principal that needs to allocate a single good among a number of agents. For example, an employer who is looking recruit a worker, a venture capital firm that is

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looking to invest in a start-up company, a government who is looking to fund a research team and so on. Each of the agents prefer to receive the good. A job applicant wants to be hired, a start-up company wants to be invested in. The valuation the principal receives from allocating the good to an agent depends on the agent's quality. The value of an employer from hiring a worker is a function of the worker's analytical and social skills, the value of a start-up company to a capital venture firm, is a function of its marketing and technological abilities. The agents have private partial information about their quality, but cannot provide evidence that ratify this information. The worker knows his social skills but cannot provide an evidence for that, a start-up company knows its technology level but may not be able provide proofs. At the starting point the principal holds the same prior beliefs about each of the agents. However, the principal can perform a costless test to each agent that provides imperfect information about the quality of the agent. For example, an employer can add to the admission process a test that roughly checks the competence of a potential applicant for the job.

We assume that the principal can commit to the properties of the test but cannot commit to an allocation.[1] That is, the principal always allocate the good to the agent who provides her the highest expected value given her information. This assumption seems rather realistic. The test is revealed to the agents and therefore it is easier to commit to, while the lack of transparency of allocation procedures makes it harder to commit on. This assumption is different than the standard assumption that appears in the papers that deal with optimal allocation with verification. In these papers the principal can commit both to a verification policy and to an allocation rule. These papers also assume that the principal can only verify information that is held by the agent, but cannot learn new information. When the principal can only verify information, the power to commit to an allocation rule is essential to the possibility of information transmission from the agents to the principal. Otherwise, each agent will transmit, independently of his true information, the message that maximizes the principal belief. In our setting a test can provide information about the agent's quality that is not held by the agent. We show that in such a case effective information transmission may arise even when the principal lacks the ability to commit to an allocation rule. We analyze

[1]We also assume that there are no transfers from the agents to the principal.

how these considerations affect the principal's optimal test choice.

We find that effective information transmission is possible only if the information agents posses is less important to the principal than the information they do not hold. We consider the implications of information transmission in two situations. In the first, the principal is facing a unique test. We analyze the principal's payoff as a function of the number of agents that participate in the allocation process. We find that the principal's utility may not be monotonic with the number of agents. That is, in some cases it is better off for the principal to examine less agents despite the fact that the tests are costless. In the second situation the principal faces a number of tests to choose from. We find that the principal will always choose a test that induces information transmission. Such a test is always different from the test she will choose in situations where information cannot be transmitted. These results imply that the principal strictly prefers that agents will know information that is less valuable for her.

2 The Model

The set of agents is $N = \{1, .., n\}$. There is a single indivisible good to allocate among them. We denote by ω_i the value the principal receives from allocating the good to agent i. This value depends on the quality of the agent. We assume that the agent holds partial information about his quality and denote this information by x_i. The information about an agent's quality that is not held by the agent is denoted by y_i. For simplicity we assume that $x_i, y_i \in \{0, 1\}$ and that the agent's value depends on these arguments in a linear manner, i.e., $\omega_i (x_i, y_i) = x_i + a \cdot y_i$ where $a \geq 0$. We assume that x_i and y_i are independently distributed according to the uniform distribution and that agents' types are independent. The agents communicate with the principal via costless unverifiable messages. The principal can also learn directly and without cost information about an agent's type by performing a monotonic binary test, t, (success or failure) of the following form

1	p	1
0	0	q
y_i/x_i	0	1

Where p and q indicate the probability of success in the relevant entries. For example the probability of type $(0,1)$ to pass the test is p. The assumption that the tests are binary is a way to restrict attention to the situation where a test cannot reveal all the information about the agents. The principal can commit to a test but cannot commit to an allocation rule. That is, the principal always allocate the good to the agent who provides him the highest expected value given his information. We assume that every agent strictly prefers receiving the object to not receiving it. Consequently, we can take the payoff to an agent to be the probability he receives the good. The intensity of the agents' preferences plays no role on the analysis so these intensities may or may not be related to the types.

3 Example of Information Transmission

In this section we illustrate by an example how effective information transmission may arise in our setting. In subsequent sections we analyze how information transmission considerations affect the optimal learning procedure of the principal. Consider $N = \{1,2,3\}$ and $\omega_i(x_i, y_i) = x_i + 3 \cdot y_i$ and assume that the principal can perform a test of the following form

1	p	1
0	0	0
y_i/x_i	0	1

The principal allocates the object to the agent with the highest expected value, and in case some of them share the highest expected value, allocates the object to each of these agents with equal probability. We characterize the values of p for which there exists a truthful equilibrium. This result relies on the communication phase taking place prior to the tests. Given that the principal believes that the agents state the truth, the expected values she assigns to an agent as a function of his message and his test result are

$$E[\omega_i|s,1] = 4, \quad E[\omega_i|s,0] = 3, \quad E[\omega_i|f,0] = \frac{1-p}{2-p} \cdot 3, \quad E[\omega_i|f,1] = 1$$

E.g., $E[\omega_i|s,1]$ is the expected value of agent i given that he succeeded in the test and that he reported $x_i = 1$. For values of p that are above $1/2$ we get that $E[\omega|f,1] > E[\omega|f,0]$. This means that irrespective of the realization of the test result it is better for the agents to report $x_i = 1$. Hence, a truthful equilibrium is impossible. Nonetheless, for values of p that are below $1/2$ it holds that $E[\omega|f,0] > E[\omega|f,1]$. That is, an agent who failed the test prefer that the principal would believe he is of type $x_i = 0$. Such an agent wants to trade-off bad valuation on x_i for good beliefs on y_i. This means that although the agent wants the principal's valuation to be as high as possible he would prefer that the principal would think he has a low value of x_i. Given that $p < 1/2$ we can calculate the probability that an agent will receive the object as a function of his statement and a test result. E.g., $(S,1)$ is the probability that an agent who succeeded in the test and reported $x_i = 1$ will receive the object.

$$(S,1) = \frac{37}{48}$$

$$(S,0) = \frac{p^2 - 9p + 27}{48}$$

$$(F,0) = \frac{p^2 - 7p + 13}{48}$$

$$(F,1) = \frac{1}{48}$$

Now, agents with different values of x_i have different probabilities of succeeding in the test, so they face different lotteries. In a truthful equilibrium each agent prefers the lottery that corresponds to his true value of x_i. The incentive compatibility constraint of an agent with $x_i = 0$ is

$$\frac{p}{2} \cdot \left(\frac{p^2 - 9p + 27}{48}\right) + \frac{2-p}{2} \cdot \left(\frac{p^2 - 7p + 13}{48}\right) \geq \frac{p}{2} \cdot \frac{37}{48} + \frac{2-p}{2} \cdot \frac{1}{48}$$

The incentive compatibility constraint of an agent with $x_i = 1$ is

$$\frac{1}{2} \cdot \left(\frac{p^2 - 9p + 27}{48}\right) + \frac{1}{2} \cdot \left(\frac{p^2 - 7p + 13}{48}\right) \leq \frac{1}{2} \cdot \frac{37}{48} + \frac{1}{2} \cdot \frac{1}{48}$$

The set of values of p for which all the above constraints are satisfied is $\left[4-\sqrt{15}, 1/2\right)$. That is, for these values there exists a truthful equilibrium.

4 Tests that Enable Information Transmission

In this section we characterize the tests that sustain a truthful equilibrium.[2] The principal examine all the agents.[3] We assume that the principal can perform a single test for all the agents. This assumption captures the property that preparing a test is costly while examining the agents is virtually costless. We assume that the principal can only commit to a test. For clarity of exposition we restrict our attention to test with $q = 0$. This restriction does not affect the results of the paper regarding the principal's optimal test selection. That is, the characterization of the optimal test remains the same also in the general model where we allow q to vary.

Each of the agents is looking to maximize his probability of receiving the object. In a truthful equilibrium a message from the agent and a realization of the agent's test result translates to a belief of the principal regarding the agent's quality via Bayes law. The principal beliefs are as follows.

$$E\left[\omega_i|s,1\right] = a+1, \quad E\left[\omega_i|s,0\right] = a, \quad E\left[\omega_i|f,0\right] = \frac{1-p}{2-p}\cdot a, \quad E\left[\omega_i|f,1\right] = 1$$

The principal allocates the object to the agent with the highest expected value, and in case some of them share the highest expected value allocates the object to each of these agents with equal probability. A truthful equilibrium can arise only if $E\left[\omega_i|f,0\right] \geq E\left[\omega_i|f,1\right]$. Otherwise, each agent would prefer tp report $x_i = 1$ irrespective of the realization of the test result. We define $\hat{p}\left(a\right) = \frac{a-2}{a-1}$ to be the maximal p such that in a truthful equilibrium we have $E\left[\omega_i|f,0\right] \geq E\left[\omega|f,1\right]$ and state the following claim

Claim 1. A truthful equilibrium exists only if $p < \hat{p}\left(a\right)$.

Since $0 \leq p$ we can also deduce the following corollary

[2]That is, we restrict our analysis to equilibiria with pure strategies. In later sections we discuss also equilibria with mixed strategies.

[3]This assumption does not preclude the case where the principal can commit to examine only a subset of agents, because even in such a case she would always choose to examine all the agents.

Corollary 2. *A truthful equilibrium exists only if $a \geq 2$.*

The interpretation of Claim 1 and corollary 2 is that for truthful equilibrium to occur both the receiver and the agent separately cannot hold too much information about the agent's quality.[4]

We now turn to find properties of tests that allow for a truthful equilibrium given that the principal allocates the object according to the following order

$$\text{(sucess,1)} \succ \text{(sucess,0)} \succ \text{(failure,0)} \succ \text{(failure,1)}$$

We calculate the probability that an agent will receive the object as a function of his statement, the number of agents, and a test result. E.g., $S1\,(n,p)$ is the probability that an agent who succeeded in the test and reported $x_i = 1$ will receive the object, given the test p and the number of agents n.

$$S1\,(n,p) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \frac{i}{i+1} \cdot \left(\frac{1}{4}\right)^{i} \cdot \left(\frac{3}{4}\right)^{n-i-1} \right)$$

$$S0\,(n,p) = 1 - \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \left(\frac{1}{4}\right)^{i} \cdot \left(\frac{3}{4}\right)^{n-i-1} \right) - \left(\sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \frac{i}{i+1} \cdot \left(\frac{p}{4}\right)^{i} \cdot \left(\frac{3-p}{4}\right)^{n-i-1} \right)$$

$$F0\,(n,p) = \sum_{i=1}^{n-1} \binom{n-1}{i} \cdot \frac{1}{i+1} \cdot \left(\frac{2-p}{4}\right)^{i} \cdot \left(\frac{1}{4}\right)^{n-i-1} + \left(\frac{1}{4}\right)^{n-1}$$

$$F1\,(n,p) = \frac{1}{n} \cdot \left(\frac{1}{4}\right)^{n-1}$$

Each agent will find it optimal to report truthfully given the principal's beliefs if and only if the following constraints hold.

$$\text{IC of } x_i\text{=0:} \quad \frac{p}{2} \cdot S0\,(n,p) + \frac{2-p}{2} \cdot F0\,(n,p) \geq \frac{p}{2} \cdot S1\,(n) + \frac{2-p}{2} \cdot F1\,(n)$$

$$\text{IC of } x_i\text{=1:} \quad \frac{1}{2} \cdot S0\,(n,p) + \frac{1}{2} \cdot F0\,(n,p) \leq \frac{1}{2} \cdot S1\,(n) + \frac{1}{2} \cdot F1\,(n)$$

[4]The larger p is the smaller the principal's expected deviation from the true state of the world given the information she receives from the test.

For every $x_i \in \{0,1\}$ we define $IC_{x_i}(n)$ to be the set of p's for which the IC constraint of type x_i holds and $p_{x_i}(n)$ to be the p for which the IC of type x_i holds in equality.

Lemma 3. *$IC_0(n) = [0, p_0(n)]$ and $IC_1(n) = [p_1(n), 1]$*

The result regarding $IC_0(n)$ follows from the fact that the expected payoff of $x_i = 0$ given a truthful report does not depend on p , while the incentive of $x_i = 0$ to misreport is increasing in p because $p_r(s|0) = \frac{p}{2}$. The result regarding $IC_1(n)$ follows from the fact that the expected payoff of $x_i = 1$ given a truthful report does not depend on p , while the incentive of $x_i = 1$ to misreport is decreasing in p because both $S0(n,p)$ and $F0(n,p)$ are decreasing in p.

Lemma 4. *$p_1(n) < p_0(n)$ for every $2 < n$.*

An agent wants to report 1 in case of a success and 0 in case of a failure irrespective of his type. The result follows from the fact that the probability of success for type $x_i = 1$ is higher than that of type $x_i = 0$. This means that for each $p \in [0,1]$ for which type $x_i = 1$ weakly prefers to misreport, type $x_i = 0$ strongly prefers to report truthfully; and for each $p \in [0,1]$ for which type $x_i = 0$ weakly prefers to misreport, type $x_i = 1$ strongly prefers to report truthfully.

We now characterize the set of tests in which a truthful equilibrium exists. These tests have to satisfy both the condition about the appropriate order, i.e., $p < \hat{p}(a)$ and the incentive compatibility conditions.

Theorem 5. *Assume $a > 2$ and $n \geq 2$. There exist two decreasing functions $p_1(n) < p_0(n)$ where $0 < p_0(n)$ and there exist $n' < n''$ such that effective cheap talk arises if and only if*

$$p \in A := [p_1(n), p_0(n)] \cap [0, \hat{p}(a)).$$

If $n \leq n'$ then A is empty. ($\hat{p}(a) \leq p_1(n')$)

If $n' < n < n''$ then $A = [p_1(n), \hat{p}(a))$

If $n'' < n$ then $A = [p_1(n), p_0(n)]$.

As n increases the probability that there is an agent of type $(s, 1)$ increases. Hence, the incentive to report $x_i = 1$ increases. This implies that $IC_0(n+1) \subset IC_0(n)$ and $IC_1(n) \subset IC_1(n+1)$. The result on the monotonicity of $p_1(n)$ and $p_0(n)$ follows from Lemma 3. At $p = 0$ the IC of $x_i = 0$ holds strictly, from continuity we get that $0 < p_0(n)$ for every $1 < n$.

5 The Implications of Information Transmission

In this section we show how the existence of a truthful equilibrium can affect the principal's preferences regarding the economic environment she is facing. We consider two situations. In the first situation there is no cheap talk communication prior to the test, while in the second situation there is. We compare between these situations in two economic environments. In the first environment the principal is facing a fixed test, while in the second environment the principal can choose a test. We find that if the principal is facing a fixed test then when there is no cheap talk communication the principal would always prefer that more agents would participate in the allocation process. However, when there is cheap talk communication the principal's payoff may not be monotonic in the number of agents. That is, for some tests the principal would prefer that less agents would participate in the allocation process despite the fact that the tests are costless. When the principal can choose the test, we find that when there is cheap talk communication the principal almost always chooses a different test than the test she chooses when there is no cheap talk communication. That is, the principal gains a strictly higher payoff if she communicates with the agents.

5.1 A Principal Facing a Fixed Test

Assume the principal can perform a single test

1	p	1
0	0	0
y/x	0	1

where $p \in [0, 1]$. Consider the situation where there is no cheap talk communication.

In this case the principal's payoff is strictly increasing in the number of agents n. That is, the larger the number of agents that participate in the allocation process the higher is the expected value of the agent who will receive the object. To see this note that the expected value of the highest ranked agent is

$$\left(1-\left(\frac{3-p}{4}\right)^{n}\right)\cdot E\left(\omega_{i}|S\right)+\left(\frac{3-p}{4}\right)^{n}\cdot E\left(\omega_{i}|F\right)$$

since $E\left(\omega_{i}|S\right)>E\left(\omega_{i}|F\right)$ this expression is increasing in n.

We now consider the case where the agents communicate with the principal prior to the test. We first show that given a fixed test p the existence of truthful equilibrium depends on the number of agents.

Claim 6. for every $p<\hat{p}\left(a\right)$ there exist $\underline{n}\left(p\right)\leq\overline{n}\left(p\right)$ such that truthful equilibrium exists if and only if $\underline{n}\left(p\right)\leq n\leq\overline{n}\left(p\right)$.

In Theorem 5 we saw that the set of test for which truthful equilibrium exists $\left[p_{1}\left(n\right),p_{0}\left(n\right)\right]$ depends on the number of agent. Fix a test p. We get that for $n<\underline{n}\left(p\right)$ we have $p<p_{1}(n)$; for $\underline{n}\left(p\right)\leq n\leq\overline{n}\left(p\right)$ we have $p_{1}(n)\leq p\leq p_{0}(n)$; and for $\overline{n}\left(p\right)<n$ we have $p_{0}(n)<p$.

We now show that the principal's payoff may not be monotonic in the number of agents. To see this consider the following example. Assume that $a=2.5$ and that the principal is facing the test $p=\frac{11}{51}$ in this case we have that $\overline{n}\left(\frac{11}{51}\right)=6$. That is, if the number of agent is larger than 6 then given the test $p=\frac{11}{51}$ there is no truthful equilibrium . The principal's payoff when $n=6$ is larger than her payoff when $n=7$. The intuition for this result is the following. The increase of n from 6 to 7 implies that a truthful equilibrium cease to exist. Without a truthful equilibrium the principal can distinguish between agents based only on the results of the test - success or failure. On the other hand, a truthful equilibrium allows the principal to distinguish between agents within these categories. Increasing the number of agents improves the expected ability of the best agent. However, the identification of the best agent is better under a truthful equilibrium.

5.2 A Principal who Chooses the Test

Assume that the principal can choose any test. That is she can choose any $p \in [0,1]$. Consider the situation where there is no cheap talk communication. We now characterize the properties of the optimal test as a function of the number of agents n.

Claim 7. For every $a > 2$ the optimal test $p(n,a)$ is a decreasing function of n and there exists $n_0(a)$ such that for every $n_0(a) \leq n$ we have $p(n,a) = 0$.

That is, in some number of agents the principal chooses the test $p = 0$. The explanation for this result is the following. As the number of agents increases the probability of the event that at least one of the agents is of the best type (1,1) increases. Therefore, the principal would want to choose the test that accurately identify type (1,1).

We now consider the case where the agents communicate with the principal prior to the test. We show that for every n we have that the optimal test $p(n,a) > 0$. That is the optimal test is different than in the case where there is no cheap talk communication.

Theorem 8. *Consider $a > 2$. For every* $\max\{n_0(a), n'\} \leq n$ *the principal will choose the maximal p that allows for truthful equilibrium, i.e., $p(n,a) = \min\{\hat{p}(a), p_0(n)\}$ which is greater than 0 for every n.*

For $n_0(a) \leq n$ the optimal test without cheap talk, $p = 0$, identifies the best type (and pool all other types). Given a test $p > 0$ that allows for a truthful equilibrium the best type (1,1) is identified and since p is positive there is also a partial identification of the second best type (0,1). That is, choosing the maximal p that allows for a truthful equilibrium dominates the test $p = 0$. This result implies that by communicating with the agents the principal can achieve a strictly higher payoff. The following table shows values of $n_0(a)$ and $p_0(n_0(a))$ for several values of a.

a	$\hat{p}(a)$	$n_0(a)$	$p_0(n_0(a))$
2.5	0.33	6	0.21
3	0.5	5	0.31
4	0.66	8	0.11

6 Conclusion

We have considered the problem of optimal allocation without transfers and with partial commitment. We showed that if both the principal and the agents hold partial information about the state of the world then effective information transmission may arise. This happens in situations when both parties do not hold too much information. We showed how these cheap talk considerations are incorporated into the optimal allocation process. Our model consist of state-independent senders. The literature on information transmission with state-independent senders has largely ignored cheap talk games and rather has focused on either signaling or disclosure games. Our model is an example of a design problem in a cheap talk model where considerations of effective information transmission arise even when agents have state independent preferences.